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The
MATHEMATICS STUDENT

Edited by

C. N. SRINIVASIENGAR, D.Sc.

With the Cooperation of

DR. HANSRAJ GUPTA, M.A., Ph.D.
DR. RAM BEHARI, M.A., Ph.D., D.Sc., F.N.I.
DR. V. RANGACHARIAR, D.Sc.
DR. K. VENKATACHALIENGAR, D.Sc.

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THE MATHEMATICS STUDENT

Vol. XVIII SEPTEMBER—DECEMBER 1950 Nos. 3 & 4

PAṬIGAṆITA AND THE HINDU ABACUS

BY

R. VENKATACHALAM IYER

The aim of this paper is to throw some fresh light on certain obscurities in the meaning of *Pāṭīgaṇita* (पाटीगणित) as expounded in the *History of Hindu Mathematics*, Part I, by Bibhuti Bhushan Datta and Avadesh Narayan Singh (1935 Edition). The following quotations are from Chapter II of their valuable contribution to the History of Hindu Mathematics.

"Arithmetic forms the major part of the Hindu works on *Pāṭīgaṇita.* The word *Pāṭīgaṇita* is a compound word formed from the words *Pāṭī* meaning 'board' and *gaṇita* meaning 'science of calculation'; hence it means the science of calculation which requires the use of writing material (the board). It is believed that the term originated in a non-Sanskrit literature of India, a vernacular of North India. The oldest Sanskrit word for the board is *Phalaka or Paṭṭa,* not *Pāṭī.*" (Page 123)

"The only works avilable which deal exclusively with *Pāṭīgaṇita* are the *Bakshāli Manuscript*—(c. 200), the *Triśatika* (c. 750), the *Gaṇita-Sāra-Sangraha* (c. 850), the *Gaṇita-tilaka* (1039), the *Līlāvatī* (1150), the *Gaṇita-Kaumudi* (1356), and the *Pāṭī-sāra* (1658)." (Page 125)

As the Sanskrit term for board is not *Pāṭī*, it is highly questionable to assume that *Pāṭī* must mean a board and postulate a North Indian Vernacular for the purpose. It is worthy of note that the compound word *Pāṭīgaṇita* does not occur in any of the works cited above, while the word *Pāṭī* occurs alone, very often, in contexts where the meaning 'board' would be quite inappropriate. Further, the assumption that the use of writing materials is necessary for calculation is not justified, in the light of the practice current among orthodox Hindu Pandits and the evidence of the words used in the stanzas regarding the procedure to carry out the fundamental operations of arithmetic.

In the absence of the word *Pāṭīgaṇita* in the texts, its meaning has to be construed from the word *Pāṭī* which occurs in them, as well as from reference to standard Sanskrit Lexicons.

The word *Pāṭī* occurs in the opening stanza * of *Līlāvatī*, *Gaṇita-tilaka* and *Triśatika*. In the context of the subject matter of each treatise, it is impossible for the word *Pāṭī* to mean 'a board.' **पाटीं सद्गणितस्य वच्मि** of *Līlāvatī* and **करोमि......गणितस्य पाटीं** of *Gaṇita-tilaka* can only be rendered, "I propound correct mathematical technique", for, it would be outrageous to translate the above words as, "I propound the board of mathematics."

In the opening stanza of *Triśatika* and in some later verses of *Līlāvatī*, the word *Pāṭī* is used as a contraction for *Gaṇitasya Pāṭī* and in all such cases the translation "mathematical technique or procedure" would be quite appropriate.

Let us now see what Sanskrit Lexicons have to say on the topic. Monier Williams in the Oxford Sanskrit Dictionary reckons *Pāṭī* as a Sanskrit word and derives it from the root **पाट्**, meaning regular order or succession. *Śabda Sāgara* gives the meaning of the word as "arithmetic", "regular order" deriving it from the root **पट्**—to go. *Sabda Sthōma Mahōdadhī* gives

पाटी—स्त्री—पट—णिच्—इ—ङीप्

धारायाम्, व्यक्तगणनायां लीलावत्यादौ

Again, *Vācaspatya* has the following

पाटी—स्त्री—पट—(स्वार्थे णिच्)—इ—ङीप्.

1. बालावृक्षे
2. परिपाट्यां गुणनादीनाम्
3. स्पष्टक्रमे च

"अस्ति त्रैराशिकं बीजं पाटी च विमला गतिः" (लीला.)

पाटीगणित—पाट्या परिपाट्या गणितम्।

लीलावात्याद्यङ्कस्वरूपे गणितशास्त्रे

"पाटी नाम सङ्कलितव्यवकलितगुणनभजनादीनां क्रमः।

तया युक्तं गणितं पाटीगणितम्"। (लीलावतीटीका)

---

* 1. प्रीतिं भक्तजनस्य यो जनयते विघ्नं विनिघ्नन् स्मृत
स्तं वृन्दारकवृन्दवन्दितपदं नत्वा मतङ्गाननम्।
पाटीं सद्गणितस्य वच्मि चतुरप्रीतिप्रदां प्रस्फुटां
संक्षिप्ताक्षरकोमलामलपदैर्लालित्यलीलावतीम्॥ *Līlāvatī*

2. रूपोज्झितं रूपयुतं स्वरूपमात्मस्वरूपं परमं प्रणम्य।
करोमि लोकव्यवहारहेतोर्विचित्रवृत्तां गणितस्य पाटीम्॥ *Gaṇita-tilaka*

नत्वा शिवं स्वविरचितपाट्या गणितस्य सारमुद्धृत्य।
लोकव्यवहाराय प्रवक्ष्यति श्रीधराचार्यः॥ *Triśatika*

In the light of these facts the word *Pāṭīgaṇita* as used by recent writers has to be taken as equivalent to *Gaṇitasya Pāṭī* or science of computation. Hence the attempt to derive the word from the board used as writing material appears unsatisfactory.

A survey of the Sanskrit works cited as dealing with *Pāṭīgaṇita* shows that they contain the whole of mathematical lore, then known. Thus **त्रैराशिक**—(rule of three or principle of proportion, the main content of the arithmetic of daily life involving calculation on the Abacus), **बीज**—(Algebra including indeterminate equations, series, permutations and combinations), and **क्षेत्रगणित**—(geometry including plane and spherical trigonometry), all belong to *Pāṭīgaṇita* and denote different techniques*, though the word connotes arithmetic in particular.

The authors of the *History of Hindu Mathematics*, Part I, observe on page 6,

" For calculations involved in *Gaṇita* the use of writing material was essential. The calculations were performed on a board (*Pāṭī*) with a piece of chalk or on sand (*Dhūli*) spread on the ground or on the *Pāṭī* ". On page 124 they observe,

" Bayley, Fleet and several others suspect that the origin of the name *Pāṭī* in Hindu Mathematics lies in the use of the board as an Abacus. This conjecture however, is without foundation, as no trace of the use of any form of the Abacus is found in India."

These observations are flatly contradicted, not only by the practice of orthodox Pandits and Jyotiṣis carrying out mathematical calculation using a flat board and cowries, without writing down any numerical figures, but also by the description of mathematical operations as given in the text books cited by the authors as dealing with *Pāṭīgaṇita*.

Recording results of calculation on paper or on cadjans using Hindu numerals is entirely different from using the numerals themselves in the process of calculation. The Hindu Pandits carry out calculations on a flat board with cowries, using no written symbols, and write down the results of calculation alone, on paper or on cadjans. During calculation, they never write figures on the board or on the ground or on paper. Also they never cut off, or rub off unwanted figures to be replaced by new figures. At any stage of working sums on the board, only a line or two of groups of cowries are to be seen

* " अस्ति त्रैराशिकं बीजं पाटी च विमला गतिः" *Līlāvatī*

" श्रीधरनारायणादिभिरपि भाण्डजात्यादिकमन्यदप्युक्तम् ।
वास्तवन्तु मिश्रादीनां त्रैराशिकगम्यत्वेन त्रैराशिकमेव पाटी " ।

Buddhi-vilāsini Tīka on Līlāvatī

there. For example, during a multiplication, only the line of groups of cowries representing the multiplicand and a line of cowries denoting the sum of the partial products obtained till that stage of the multiplication are to be seen on the board, in addition to the cowries representing those digits only of the multiplier still pending operation. This being the case at present, during the twentieth century, when writing materials are cheaply available, the contention that figures were written on the board in ancient days and changed one by one by cutting off, erasure, or substitution, is entirely misleading. The fact that orthodox Pandits carry out calculations on a flat board with cowries even now, as a matter of tradition, contradicts the contention that there is no trace of the use of any form of the Abacus in India, however much one would like to think that our ancestors of 250 A.D. were carrying out calculations with written numerals.

The suggestion that figures were not written on the board during calculation in ancient days is also supported by the description of the procedure given in all ancient Hindu Mathematical works. In their rules of procedure the ancient authors only use words which mean "place the number", "move the number", "take away the number" "remove the number" and so on. They never use words which mean "write the number" "rub off the figure", "cut off the digit", "substitute this digit in place of the one cut off", and so on. The words to be found in rules of operation (करणसूत्र) are न्यासः, न्यसेत्, स्थाप्य, क्षिपेत्, त्यक्त्वा, उत्सार्य, etc. If writing down a number was intended, they could have used लिखेत् in place of न्यसेत् without destroying the metre. The fact that they did not do so is sufficient evidence that numerals were not written by them on the board.

In this connection, it should also be noted that the quotation given by the authors of the *History of Hindu Mathematics* on page 129 from Bhaskara II—namely *Khaṭikāya rēkhē ucchādya*—(having drawn lines with chalk)—does not contradict the above suggestion, since the quotation is taken from the chapter on eclipses and only refers to the drawing of diagrams (परिलेखनानि) with no reference to calculation. A demonstration of indigenous calculation with cowries on a board will show that the details of performing fundamental operations as presented in the *History of Hindu Mathematics* is a caricature of the actual process.

In conclusion, it is suggested that *Pāṭīgaṇita* means "Practical mathematics" or "Technique of calculation," and that, though the word *Pāṭī* does not mean a board, the methods described in *Pāṭīgaṇita* relate to calculation with cowries on the board or the Hindu Abacus.

THE FUNCTION $\tau(n)$ OF S. RAMANUJAN

(AN EXPOSITORY LECTURE *)

BY

F. VAN DER BLIJ, *St. Laurenslaan 2, Breda, Netherlands*

1. Introduction, congruences.

The function $\tau(n)$ can be defined for every positive integer n by the following identity

$$\sum_{1}^{\infty} \tau(n)\, x^n = x \prod_{1}^{\infty} (1-x^n)^{24}.$$

The infinite product converges if $|x| < 1$. But why did we choose the power 24? Hardy [3] says: "We may seem to be straying into one of the backwaters of mathematics, but the genesis of $\tau(n)$ as a coefficient in so fundamental a function compels us to treat it with respect". Which fundamental function is meant? Well, it is the number of representations of a number n as a sum of 24 squares. Furthermore we shall see that many principal lines in the theory of quadratic forms, θ-functions and Dirichlet series, in the theory of elliptic invariants, modular forms and functions lead to these numbers $\tau(n)$.

To mention one example we formulate a hypothesis, quite analogous to the Riemann-hypothesis for the ζ function.

Ramanujan [1] calculated a table of values of $\tau(n)$ for $n \leqslant 30$, Lehmer & Gupta extended it to $n=400$, later on Watson [2] published a table of values of $\tau(n)$ for $n \leqslant 1000$. Recently Lehmer has extended the $\tau(n)$ table up to $n=2500$. We write some values down

$$\tau(1)=1, \quad \tau(2)=-24, \quad \tau(3)=252, \quad \tau(4)=-1472$$
$$\tau(5)=4830, \quad \tau(6)=-6048, \quad \tau(7)=-1674.$$

In the following proofs we need the well-known identities of

Euler: $$\prod_{1}^{\infty}(1-x^n) = \sum_{-\infty}^{+\infty} (-1)^n\, x^{\frac{1}{2}n(3n+1)}$$

Jacobi: $$\prod_{1}^{\infty}(1-x^n)^3 = \sum_{0}^{\infty} (-1)^n (2n+1)\, x^{\frac{1}{2}n(n+1)}$$

$$= x^{-\frac{1}{8}} \sum_{-\infty}^{+\infty} (-1)^n \left(n+\tfrac{1}{2}\right) x^{\frac{1}{2}(n+\frac{1}{2})^2}$$

Legendre: $$\left[\sum_{0}^{\infty} x^{\frac{1}{2}n(n+1)}\right]^4 = \sum_{0}^{\infty} \sigma(2n+1)\, x^n.$$

Here $\sigma(k)$ denotes the sum of the positive divisors of k;

$$\sigma(k) = \sum_{d|k} d.$$

---

* Somewhat enriched translation of *Mathem. Centrum* ZW – 010 (1948)

Using these identities we are able to prove some congruence properties of the function $\tau(n)$.

(i) $\tau(2k+1) \equiv \sigma(2k+1) \pmod 8$, $\tau(2k) \equiv 0 \pmod 8$.

(Bambah, Chowla, Gupta [3]).

Starting from $(1-x)^8 \equiv (1-x^2)^4 \pmod 8$ that is to say

$$(1-x)^8 = (1-x^2)^4 + 8y,$$

where y is a polynomial (power series) with integral coefficients—we obtain the formula

$$\sum \tau(n)\, x^n \equiv x\,\{(1-x^2)(1-x)^4 \cdot \cdot \cdot \cdot\}^{12} \pmod 8,$$

$$\equiv x \left\{\sum_0^\infty (-1)^n (2n+1)\, x^{n(n+1)}\right\}^4 \pmod 8,$$

$$\equiv x \left\{\sum_0^\infty x^{n(n+1)}\right\}^4 \pmod 8,$$

$$\equiv x \sum_0^\infty \sigma(2k+1)\, x^{2k} \pmod 8,$$

$$\equiv \sum_0^\infty \sigma(2k+1)\, x^{2k+1} \pmod 8.$$

(ii) $\tau(7m+k) \equiv 0 \pmod 7$, if $\left(\frac{k}{7}\right) = -1$, that is to say if $k = 3,5,6$ (Hardy [3]). We write

$$\sum \tau(n)\, x^n = x\,\Pi(1-x^n)^{21}\ \Pi(1-x^n)^3.$$

We have $(1-x^n)^7 \equiv (1-x^{7n}) \pmod 7$,

thus $(1-x^n)^{21} \equiv 1 - 3x^{7n} + 3x^{14n} - x^{21n} \pmod 7$,

and $\Pi(1-x^n)^{21} = \sum c_\mu x^\mu$ where $\mu\, c_\mu \equiv 0 \pmod 7$.

Using the identity of Jacobi we obtain

$$\sum \tau(n)\, x^n = \sum_{\mu=0}^\infty \sum_{\nu=0}^\infty (-1)^\nu (2\nu+1)\, c_\mu\, x^{\frac{1}{2}\nu(\nu+1)+\mu+1}$$

thus
$$\tau(n) = \sum (-1)^\nu (2\nu+1)\, c_\mu,$$

where the summation must be extended over all positive integers μ and ν satisfying $n = \frac{1}{2}\nu(\nu+1)+\mu+1$. Since $\frac{1}{2}\nu(\nu+1) \equiv 0,1,3,6 \pmod 7$ we find $n-\mu \equiv 0,1,2,4 \pmod 7$. If $n \equiv 3,5,6 \pmod 7$ we have $\mu \not\equiv 0 \pmod 7$ and so we have proved that every occurring c_μ is a multiple of seven.

(iii) Starting from

$$\sum \tau(n)\, x^n = x\,\Pi(1-x^n)^{23}\ \Pi(1-x^n),$$

we can prove $\tau(23n+k) \equiv 0 \pmod{23}$, if $\left(\frac{k}{23}\right) = -1$.

(iv) We mention some other congruences $\left(\sigma_k(n) = \sum_{d|n} d^k\right)$:

$\tau(n) \equiv \sigma_{11}(n) \pmod{2^8}$, n odd, Bambah, Chowla [7]

$\tau(n) \equiv \sigma_3(n) \pmod{2^5}$, n odd, Bambah, [1]

$\tau(n) \equiv (n^2+k)\,\sigma_7(n) \pmod{3^4}$, $k=0$ if $n \equiv 0, 1 \pmod 3$, $k=9$ if $n \equiv 2 \pmod 3$, Bambah, Chowla [7]

$\tau(n) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{5^3}$, $(n, 5)=1$, Bambah, Chowla [7]

$\tau(n) \equiv n^2\sigma(n) \pmod{3^2}$, Bambah, Chowla [6]

$\tau(n) \equiv n\,\sigma_9(n) \pmod{5^2}$, Bambah [1]

$\tau(n) \equiv n\,\sigma_3(n) \pmod 7$, Gupta [3], Ramanathan [2], Wilton [2]

$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$, Ramanujan, [1], Lehmer [1].

Lahiri published a list of 171 congruences, deduced systematically from identities of the theory of modular forms (compare Ramanujan [1]). We give two items as an example

$$\tau(n) \equiv n^2\,\sigma_7(n) \pmod{2^2\cdot 3^3\cdot 5}$$

$$5189184\,\tau(n) \equiv -11\cdot 691^2\,\sigma_{13}(n) + 2730\,(11n+2719)\,\sigma_{11}(n) - 691\,[6006\,(65n-5)\,\sigma_9(n) - 10331\sigma(n)] \pmod{2^7\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 691}.$$

We can extract the following essential congruences (Lehmer [4])

$$\tau(n) \equiv (175n^2+499n)\,\sigma_7(n) + (473n^3+235n^2-157n)\,\sigma_5(n) + (224n^4-390n^3+155n^2+473n)\,\sigma_3(n) + (48n^5+76n^4+396n^3-53n^2-105n)\,\sigma(n) \pmod{2^{10}}$$

$$\tau(n) \equiv n^2\,\sigma_7(n) - 18n^2\sigma_5(n) + 9n^2(2n+5)\sigma_3(n) - 45n^3\sigma(n) \pmod{3^5}$$

$$\tau(n) \equiv 36n\sigma_9(n) - 30n\sigma_5(n) - 40n^2\sigma_3(n) + 35n\sigma(n) \pmod{5^3}$$

$$\tau(n) \equiv n\sigma_3(n) \pmod 7$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

A numerical calculation of Ferguson proved that the 52nd coefficient in the power series of $\Pi(1-x^n)^{15}$ is the first one which vanishes. Now we may ask if there exists an integer N such that $\tau(N)=0$. Lehmer [3, 5] proved that the least integer N with $\tau(N)=0$ must be a prime $\geqslant 214, 928, 639, 999$. He obtained this result using the congruences; e.g. $\tau(n) \equiv \sigma_{11}(n) \pmod{2^8}$ if $\tau(N)=0$, we have $N^{11}+1 \equiv 0 \pmod{2^8}$, thus $N+1 \equiv 0 \pmod{2^8}$ and since N must be a prime we find $N \geqslant 1279$. There is little hope that it can be proved with these methods that $\tau(n) \neq 0$ for every n.

2. Modular properties.

Now we write $x = e^{2\pi i\tau}$ and we suppose $\mathrm{Im}(\tau) > 0$. With the numbers $\tau(n)$ of Ramanujan are closely connected the power series

$$\sum \tau(n)\,x^n = x\left[\prod_{1}^{\infty}(1-x^n)\right]^{24}$$

and the Dirichlet series (Ramanujan [1], Mordell [1])

$$\sum \frac{\tau(n)}{n^s} = \prod_p (1 - \tau(p)\, p^{-s} + p^{11-2s}).$$

The function

$$\Delta(\tau) = e^{2\pi i\tau} \left[\prod_1^\infty (1 - e^{2\pi i n\tau}) \right]^{24}$$

is connected with the elliptic invariants

$$g_2 = \frac{4\pi^4}{3} \left[1 + 240 \sum \sigma_3(n)\, x^n\right] = 60 \sum_{n,m}{}' (m + n\tau)^{-4},$$

$$g_3 = \frac{8\pi^6}{27} \left[1 - 504 \sum \sigma_5(n)\, x^n\right] = 140 \sum_{n,m}{}' (m + n\tau)^{-6},$$

by the formula

$$\Delta(\tau) = (2\pi)^{-12} (g_2^3 - 27\, g_3^2).$$

From this identity the behaviour of the function $\Delta(\tau)$ under the generators of the group of modular substitutions can be deduced:

$$\Delta(\tau + 1) = \Delta(\tau), \qquad \Delta\left(\frac{-1}{\tau}\right) = \tau^{12}\, \Delta(\tau).$$

We find for an arbitrary modular substitution,

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12}\, \Delta(\tau), \quad (ad - bc = 1),$$

that is to say the function $\Delta(\tau)$ is a modular form of dimension -12.

[An analytic one-valued function $f(\tau)$, which is apart from a number of poles regular in the halfplane $\operatorname{Im}(\tau) > 0$ is called a modular form of dimension k if for every integral substitution with $ad - bc = 1$, we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-k} f(\tau).]$$

From the last formula, it can be proved (Mordell [1]) that

$$\tau(mn) = \tau(m)\cdot\tau(n) \quad \text{if} \quad (m,n) = 1,$$

and $\tau(p^\alpha) = \tau(p)\, \tau(p^{\alpha-1}) - p^{11}\, \tau(p^{\alpha-2})$.

Introducing $\cos\theta_p = \tfrac{1}{2} p^{-\frac{11}{2}}\, \tau(p)$,

we may write

$$\tau(p^\alpha) = p^{\frac{11}{2}\alpha} \frac{\sin(\alpha + 1)\,\theta_p}{\sin\theta_p}.$$

Now we may ask if θ_p is a real number. We should like to prove $|\tau(p)| \leq 2p^{\frac{11}{2}}$, the best result obtained in the direction of this conjecture of Ramanujan is $\tau(n) = O(n^{\frac{23}{4}} + \varepsilon)$.

It may be seen readily from

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12}\, \Delta(\tau),$$

that the behaviour of the function $\Delta(\tau)$ at a point $\tau = p/q$ with integral p and q is closely connected with the behaviour at the point $\tau = 0$. If we design the fundamental regions of the modular group, we remark that peaks of these regions end in these points. A modular form is called a cusp form, if the constant c_0 in the Fourier expansion of

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_0^\infty c_n e^{2\pi i \tau n}$$

vanishes for every modular substitution. The theory of these cusp forms an important part of the theory of modular forms. We mention some theorems without proofs. (Hecke [4, 5])

By subtracting a suitably chosen Eisenstein series from a modular form of even dimension, we can always construct a cusp form.

There exists only one Eisenstein series of even dimension $-k < -2$,

$$G_k(\tau) = \sum_{m,n}{}' (m\tau+n)^{-k} = \alpha_k + \beta_k \sum_1^\infty \sigma_{k-1}(n)\, x^n,$$

with $\alpha_k = 2\zeta(k)$; $\beta_k = 2\,\dfrac{(2\pi i)^k}{\Gamma(k)}$.

There exist no modular forms of dimension -2 or $-(2k+1)$. If $k \equiv 2 \pmod{12}$ there are $[k/12]$, and if $k \not\equiv 2 \pmod{12}$ there are $[k/12]+1$ linearly independent modular forms of dimension $-k$. (Feldmann [1], Hecke [5], Peterson [3]).

It may be readily seen that we can deduce from these theorems the fact that there cannot exist cusp forms of even dimension $-k > -12$. In these cases every modular form equals, apart from a constant factor, an Eisenstein series. There exists only one cusp form of dimension -12. It is the well known function $\Delta(\tau)$.

So we have found at last a more comprehensible definition: *The numbers $\tau(n)$ are the coefficients of the cusp form of dimension* -12. There exist no cusp forms of dimension -14, one of dimension -16, -18, and two of dimension -24.

3. The Fourier coefficients of modular forms.

We commence with a remark concerning modular forms of positive dimension, although this has little to do with the function of Ramanujan. A well known example of a modular form of positive dimension is connected with the number of partitions of an integer n. We have $\sum p(n)\, x^n = \prod_1^\infty (1-x^n)^{-1}$

The function $\eta(\tau) = e^{\frac{\pi i \tau}{24}} \prod_{1}^{\infty} (1 - e^{2\pi i \tau})$

satisfies the function relation

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon\ (c\tau + d)^{\frac{1}{2}}\, \eta(\tau).$$

Here ε denotes a root of unity depending only on a, b, c and d.

Rademacher [4, 5] deduced the following convergent series for the coefficients $p(n)$ of the modular form of dimension ½:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{1}^{\infty} A_k(n)\, \sqrt{k}\, \frac{d}{dn} \left\{ \frac{\sinh \dfrac{C\sqrt{n - \frac{1}{24}}}{k}}{\sqrt{n - \frac{1}{24}}} \right\};$$

$$C = \tfrac{1}{3}\pi\sqrt{6},\quad A_k(n) = \sum_{h \bmod k}{}' \, w_{hk}\, e^{-\frac{2\pi i h n}{k}}\,;\quad w_{hk}^{\ 24} = 1.$$

An explicit formula for w_{hk} can be found in Hardy-Ramanujan [4] and Rademacher [3]. Rademacher proved the formula starting from an asymptotic formula for $p(n)$ of Hardy-Ramanujan, which was proved by using Farey fractions and Tauberian theorems. J. Lehner [1] deduced convergent series for the numbers $p_1(n)$ and $p_2(n)$, the number of partitions in positive integers $5l \pm 1$ and $5l \pm 2$ respectively.

A review of recent publications concerning partitions can be found by Kloosterman [2], here the congruence properties of $p(n)$ of Ramanujan are also mentioned. (Compare the non-Ramanujan* congruences of Lahiri [4, 5]).

Of a more general form are theorems of Rademacher and Zuckerman [9] on modular forms of positive dimension. Under very general suppositions a convergent series is found for the Fourier coefficient of the modular form $f(\tau)$. The series are formed by terms, which are combinations of Kloosterman sums and Bessel functions $I_{\tau+1}$.

The modular forms of dimension zero are studied by Petersson [1] starting from forms of negative dimension, by Rademacher [6, 7] starting from forms of positive dimension. The absolute modular invariant (modular form of dimension zero)

$$12^3\, J(\tau) = \frac{[1 + 240 \sum \sigma_3(n)\, x^n]^3}{x \prod (1 - x^n)^{24}} = e^{-2\pi i \tau} + \sum_{0}^{\infty} c_n\, e^{2\pi i n \tau}$$

* $p(49\, m + k) \equiv 0 \pmod{49}$ if $k = 19, 33, 40$
$p(125\, m + k) \equiv 0 \pmod{125}$ if $k = 74, 124$.

has coefficients c_n for which we have the expansion, $n \geqslant 1$

$$c_n = \frac{2\pi}{(n)^{1/2}} \sum_{1}^{\infty} \frac{A_k(n)}{k} I_1\left(\frac{4\pi\, n^{1/2}}{k}\right),$$

here $A_k(n)$ is a Kloosterman sum

$$A_k(n) = \sum_{h \bmod k}{}' e^{\frac{2\pi i}{k}(nh+h')}, \quad hh' \equiv -1 \pmod{k}$$

and $c_0 = 744$.

Congruence properties concerning the coefficients c_n are deduced by J. Lehner [2, 3].

Now we return to modular forms of negative dimension. In this case we deduce estimations for the Fourier coefficients c_n. In 1927 Hecke [1] proved that the coefficients c_n of a modular form of dimension $-k$ are of the form $c_n = n^{k-1} S(n) + O(n^{1/2 k})$, $(k \geqslant 1)$. Here $S(n)$ denotes the sum of the singular series of the modular form. In particular, we consider cusp forms. Hecke proved in 1927 for the Fourier coefficients c_n of a cusp form the estimation $c_n = O(n^{1/2 k})$, Kloosterman [1] proved in 1927 $c_n = O(n^{1/2 k - 1/8 + \varepsilon})$.
Davenport [1] and Salie [1] in 1933 that $c_n = O(n^{1/2 k - 1/6 + \varepsilon})$.
Rankin proved in 1939 that $c_n = O(n^{1/2 k - 1/5})$.

We consider the numbers $\tau(n)$. First we give the trivial estimation $\tau(n) = O(n^7)$. It follows from the definition that

$$\sum \tau(n)\, x^n = x\left[\sum_{0}^{\infty} (-1)^n (2n+1)\, x^{1/2 n(n+1)}\right]^8,$$

thus $\tau(N) = \sum (-1)^{n_1+n_2+\cdots+n_8} (2n_1+1)(2n_2+1)\cdots(2n_8+1)$,
where the summation must be extended over all integral solutions $n_1, n_2, \cdots, n_8$ of $(n_1+1/2)^2 + \cdots + (n_8+1/2)^2 = 2N$.
We refer to the well-known fact that the number of representations of N by sums of eight squares is of order $\sigma_3(N)$. Since

$$(n_i + 1/2) \leqslant (2N)^{1/2}$$

we obtain $\tau(N) = O[N^4 \sigma_3(N)] = O(N^7)$.
If we use, however, that the numbers $\tau(n)$ are the coefficients of a cusp form of dimension -12, we can obtain much better results. Hardy proved in 1918 that $\tau(n) = O(n^6)$. It follows from the estimation of Rankin [1] that $\tau(n) = O(n^{29/5})$. The same result was announced by A. Selberg [2] without proof, however. Mr. Rankin was so kind as to direct my attention to a formula

$$\tau(n) = a(1, n)/a(1, 1) = a(m, n)/a(m, 1),$$

where $$a(m, n) = 2\delta_{m,n} + 4\pi\left(\frac{n}{m}\right)^{\frac{11}{2}} \sum_{q=1}^{\infty} \frac{S(m, n;\, q)}{q} J_{11}\left(\frac{4\pi (mn)^{1/2}}{q}\right).$$

Here $\delta_{m,n}$ is Kronecker's symbol and

$$S\ (m,n\ ;\ q) = \sum_{h=1}^{q} e^{\frac{2\pi i}{2}\,(mh+n\bar{h})}\ ,\ (h,\ q)=1,\ h\bar{h}\equiv 1\ (\text{mod}\ q).$$

This formula of Petersson [1] is also proved by A. Selberg [1] and in the thesis of Rankin. It follows now from the well known estimation for the Kloosterman sums given by A. Weil [1]

$$|S\ (m,n\ ;\ q)| < 2p^{1/2}$$

that

$$\tau\ (n) = O\ (n^{6-1/4+\varepsilon}).$$

Similar results can be obtained for the coefficients of other cusp forms.

The conjecture $\tau\ (n) = O\ (n^{\frac{11}{2}+\varepsilon})$ of Ramanujan is still unproved. An estimation $c_n = O\ (n^{1/2 k - 1/2 + \varepsilon})$ for the coefficients of a cusp form could not be proved up till now.

If we denote $T\ (x) = \sum_{n \leqslant x}{}' \tau\ (n)$, where the dash means that for integral values of x, the last term must be halved, we have

$$T\ (x) = x^6 \sum_{1}^{\infty} \frac{\tau\ (n)}{n^6}\ J_{12}\ \left\{4\pi\ (nx)^{1/2}\right\}.$$

Hardy [2], Rankin [1] proved $T\ (x) = O\ (x^{\frac{59}{10}})$. It is also proved that the estimation $T\ (x) = O\ (x^{\frac{23}{4}})$ is not true. The conjecture $T(x) = O(x^{\frac{35}{6}} + \varepsilon)$ could not be proved. Finally we mention another result of Rankin:

$$\tau\ (1)^2 + \tau\ (2)^2 + \cdot\ \cdot + \tau\ (n)^2 = \alpha n^{12} + O\ (n^{12-2/5}).$$

4. Dirichlet series.

From the theory of the behaviour of modular forms under operators T_n, we recall the theorems 16, 24, 27, 28 of Hecke [4] (compare Petersson [3]). Consider the Dirichlet series $\varphi(s) = \sum_{1}^{\infty} c_n n^{-s}$, which satisfy for fixed $k \geqslant 4$ the conditions:

(i) $(s-k)\ \varphi(s)$ is an integral function of finite order.

(ii) $\varphi(s)$ is a solution of the functional equation

$$R(s) = (-1)^{1/2 k}\ R(k-s),$$

where $R(s) = (2\pi)^{-s}\ \Gamma(s)\ \varphi(s)$.

There exists always a function $\varphi(s)$, e.g.

$$\varphi(s) = \sum_{1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} = \zeta(s)\ \zeta(s-k+1).$$

This function has a simple pole at $s=k$.

The maximal number of linearly independent functions $\varphi(s)$ equals $\left[\frac{k}{12}\right]$, if $k \equiv 2\ (\text{mod}\ 12)$ and $\left[\frac{k}{12}\right]+1$ otherwise. If $k=12$ or $k \geqslant 16$ there

exist two functions $\varphi(s)$, one of which can be supposed to be regular at the point $s=k$ and it can be written as an Euler product

$$\varphi(s)=\prod_{p}\left[1-c_p\,p^{-s}+p^{k-1-2s}\right]^{-1}=\sum_{n}\frac{c_n}{n^s}.$$

The Dirichlet series corresponding to the functions

$$\Delta\ (\tau),\ \Delta\ (\tau)\ G_i\ (\tau)\ (i=4,\ 6,\ 8,\ 10,\ 14)\ ;\ G_i=\sum_{m,n}{}'\ (m\tau+n)^{-i}$$

have an Euler product of the form mentioned above. The special case

$$\sum\ \frac{\tau\ (n)}{n^s}=\prod_{p}[1-\tau(p)p^{-s}+p^{11-2s}]^{-1}$$

was supposed by Ramanujan [1]. and proved by Mordell [1].

We can construct two modular forms, linear combinations of $\Delta^2\ (\tau)$ and $\Delta\ (\tau)\ G_{12}\ (\tau)$ in such a way that the corresponding Dirichlet series can be expanded as an Euler product. The coefficients of the combinations are numbers of K ($\sqrt{144169}$), the field which can be obtained from the field of rational numbers by adjunction of the square root of the prime 144169 (Hecke [4, 5]).

It may be readily seen using

$$\sum\ \frac{c\ (n)}{n^s}=\prod_{p}\ [1-c\ (p)\ p^{-s}+p^{k-1-2s}]^{-1},$$

that $\quad c\ (mn)=c\ (m)\cdot c\ (n)$ if $(m,\ n)=1$

and $\quad c\ (p^{a})=c\ (p)\ c\ (p^{a-1})+p^{k-1}\ c\ (p^{a-2})$.

In a slightly different form

$$c\ (n)\cdot c\ (m)=\sum_{d|(n,m)}\ c\left(\frac{nm}{d^2}\right)d^{k-1}.$$

The multiplicative property of τ (n) can also be proved using the formula

$$\tau\ (n)=\frac{(-1)^{n-1}}{560}\sum\ (48s_8-184s_6s_2+146s_4^{\,2}+28s_4s_2^{\,3}-3s_2^{\,4})$$

$$n_1^{\,2}+n_2^{\,2}+\cdots+n_8^{\,2}=n,\quad s_r=n_1^{\,r}+n_2^{\,r}+\cdots+n_8^{\,r}.$$

Mr. Rankin told me that this proof was carried through in some of his papers [2, 3].

The analytic function $\varphi\ (s)=\sum\ \dfrac{\tau\ (n)}{n^s}$ satisfies the relation

$$\Gamma\ (s)\ \varphi\ (s)=(2\pi)^{2s-12}\ \Gamma\ (12-s)\ \varphi\ (12-s).$$

The Dirichlet series converges if R (s) $>\frac{13}{2}$. This follows from $\sum_{1}^{x}\tau^2\ (n)=O\ (x^{12})$. It is clear that φ (s) has no zero's with R (s) $>\frac{13}{2}$, thus all zero's, apart of trivial ones lie in the region $\frac{11}{2}\leqslant$ R (s) $\leqslant\frac{13}{2}$. We can still prove that there are no zero's with R (S) $=\frac{11}{2}$ or $\frac{13}{2}$ (Rankin [1]). The analogue of the Riemann hypothesis, that all zero's are lying on the line R (S) $=6$ could not be proved. Finally we write

down the formula: (Hardy [3]) (Compare D. H. Lehmer: *Math. Rev.* 10 (1949) page 514)

$$2^{-\frac{1}{2}} s^{-1} \sum_{1}^{\infty} \tau(n)\, e^{-2sn^{\frac{1}{2}}} = (2\pi)^{\frac{23}{2}}\ \Gamma\left(\frac{25}{2}\right) \sum_{1}^{\infty} \frac{\tau(n)}{(s^2+4n\pi^2)^{\frac{25}{2}}}.$$

5. Even quadratic forms of determinant unity.

Let $Q(x_1, x_2, \ldots, x_f)$ be a quadratic form $q_{11}x_1^2 + q_{12}x_1x_2 + \cdots$ with integral coefficients q_{ij}. We define a theta series by

$$\theta(Q, \tau) = \sum_{m_1, \cdots, m_f} e^{2\pi i \tau\, Q(m_1, \cdots, m_f)}.$$

The even quadratic form $2\,Q(m_1, \ldots, m_f)$ can be written

$$a_{11}m_1^2 + 2a_{12}m_1m_2 + a_{22}m_2^2 + \cdots$$

where the numbers a_{ij} are integers and a_{ii} are even. The behaviour of these theta functions under modular substitutions has been studied by Schoeneberg [1] and Kloosterman [3]. We can find for every even quadratic form an integer N such that the theta function is a modular form of dimension $-\frac{1}{2}f$ under all modular substitutions

$$\begin{pmatrix} ab \\ cd \end{pmatrix} \equiv \begin{pmatrix} 10 \\ 01 \end{pmatrix} \pmod{N}.$$

The least number N with this property is called the grade (Stufe) of the theta functions. This grade equals the greatest elementary divisor of the matrix $A = (a_{ij})$. Of special importance are those quadratic forms for which $|A| = N = 1$. From these forms we deduce modular forms of dimension $-\frac{1}{2}f$.

A necessary and sufficient condition for the existence of even quadratic forms is given by $f \equiv 0 \pmod 8$. We give the well-known example of Korkine and Zolotareff [1];

$$Q_8 = \sum_{1}^{8} x_r^2 + \left(\sum_{1}^{8} x_r\right)^2 - 2x_1x_2 - 2x_2x_8.$$

All even quadratic forms with 8 variables and determinant 1 can be obtained from this one by integral substitutions (they form one class of forms). There exist two classes of even quadratic forms of determinant 1 with 16 variables. The number of representations of an integer n by the form Q_8 equals $240\,\sigma_3(n)$; in the case of representation by a form Q_{16}, this number equals $480\,\sigma_7(n)$. We consider the number of solutions of the Diophantic equation $n = Q_{8l}(x_1, \cdots, x_{8l})$; that is the number of representations of $2n$ by an even quadratic form of determinant 1.

The function $\theta(\tau, Q_{8l})$ is a modular form of dimension $-4l$. Subtracting an Eisenstein series from the theta series we obtain a cusp form:

$$\theta(\tau, Q_{8l}) - \frac{(2\pi)^{4l}}{\Gamma(4l)\zeta(4l)} G_{4l}(\tau) = \theta(\tau, Q_{8l}) - 1 - \frac{(2\pi)^{4l}}{\Gamma(4l)\zeta(4l)} \sum_{1}^{\infty} \sigma_{4l-1}(n) x^n.$$

We remember that there exists no cusp form of dimension > -12. If we take $l=1$ or 2 in the formula above, we obtain

$$\theta(\tau, Q_8) = 1 + \frac{(2\pi)^{4l}}{\Gamma(4l)\zeta(4l)} \sum_{1}^{\infty} \sigma_{4l-1}(n) x^n.$$

From this identity we deduce at once the number of representations announced above. If we take $l=3$, there exists one cusp form of dimension -12, the function $\Delta(\tau)$. If $2Q_{24}$ is an even quadratic form of determinant 1 with 24 variables, we obtain for the number $Q(n, Q_{24})$ of solutions of $n = Q_{24}(x_1, \cdot\cdot, x_{24})$ the formula

$$a(n, Q_{24}) = \frac{65520}{691} \sigma_{11}(n) + c \cdot \tau(n).$$

We take as a special case

$$Q_{24}^* = Q_8(x_1, \ldots, x_8) + Q_8(x_9, \ldots, x_{16}) + Q_8(x_{17}, \ldots, x_{24}).$$

If we substitute $n=1$ we get

$$3 \cdot 240 = \frac{65520}{691} + c.$$

Finally we have $a(n, Q_{24}^*) = \dfrac{65520}{691} \sigma_{11}(n) + \dfrac{432000}{691} \tau(n).$

It can be seen from this that $65520\, \sigma_{11}(n) + 432000\, \tau(n) \equiv 0 \pmod{691}$, *thus* $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

We can deduce from a general theory of Siegel [1] that there exist two even forms $2\, Q_{24}^{**}$ and $2\, Q_{24}^{***}$ of determinant 1, in such a way that their theta series are linearly independent and linearly equivalent with $\Delta(\tau)$ and $G_{12}(\tau)$. If we denote by $a_{24}^{**}(n)$ and $a_{24}^{***}(n)$ the corresponding numbers of representations, we can thus find numbers λ, μ, ν, x such that

$$\tau(n) = \lambda\, a_{24}^{**}(n) + \mu a_{24}^{***}(n),$$
$$\sigma_{11}(n) = x\, a_{24}^{**}(n) + \nu a_{24}^{***}(n).$$

6. Some remarks.

I. We can also study the theta function defined by

$$\theta(\tau, P_k, Q) = \sum_{n_1, \ldots, n_f} P_k(n_1, \cdot\cdot, n_f)\, e^{2\pi i \tau Q(n_1, \ldots, n_f)}$$

where $P_k(x_1, \cdot\cdot, x_f)$ denotes a homogeneous polynomial of degree k which satisfies the differential equation

$$\sum_{i,j} A_{ij} \frac{\partial^2 P}{\partial x_i \partial x_j} = 0,$$

where the matrix (A_{ij}) denotes the inverse matrix of $A = (a_{ij})$.

These theta functions are modular forms of dimension $-\frac{1}{2}f-k$ and of a certain grade. If we take the even quadratic form $2\varphi_8$ with determinant 1 and $k=4$, we get a function $\theta(\tau, P_8, Q_8)$ which is a modular form of dimension -12. It can be proved that the theta functions $\theta(\tau, P_k, Q_f)$ if $k \geqslant 1$ are necessarily cusp forms. Thus we have $\theta(\tau, P_8, Q_8) = \lambda \Delta(\tau)$ and since $\lambda \neq 0$ we can find a polynomial P_8 such that

$$\tau(n) = \sum_{Q_8(m_1, \ldots, m_8)=n} P_8(m_1, \cdots, m_8).$$

This may be written by a somewhat more explicit formula. If we denote

$$2\, Q_8(x) = \sum_{r,s=1}^{8} a_{rs}\, x_r\, x_s \text{ and } A_8(x, \alpha) = \sum_{r,s=1}^{8} a_r\, x_r\, \alpha_s;\ Q_8(\alpha_i) = 1,$$

$$\tau(n) = \frac{1}{144}\left[\sum_{Q_8(m)=n} A_8(m, \alpha) - 480\, n^4\, \sigma_3(n)\right],$$

(Hecke [5] p. 86–89).

II. We mention one result concerning modular forms of grade 4. Let $\Gamma_{24}(n)$ denote the number of integral solutions of $\sum_1^{24} x_i^2 = n$. Then

$$\Gamma_{24}(n) = \frac{16}{691}\sigma_{11}^*(n) + \frac{128}{691}\{(-1)^{n-1}\, 259\, \tau(n) - 512\, \tau(\tfrac{1}{2}n)\},$$

where $\tau(\frac{1}{2}n)$ vanishes if n is an odd integer and

$$\sigma_{11}^*(n) = (-1)^n \sum_{d/n} (-1)^d\, d^{11}.$$

(Ramanujan [1], Hardy [3].)

III. Let 2Q be an even form of 24 variables with determinant q^2, and grade q. (q denotes an odd prime). Let Q^* denote the primitive adjoint form of Q, that is to say the adjoint form divided by the g.c.d. of the coefficients. Then we have (Hecke [5] p. 95):

$$q^{l-1}\, a(m, Q) + a(mq, Q^*) = (1+q^{l-1})\frac{65520}{691}\sigma_{11}(m) + a_1\, \tau(m),$$

$$q^{11-l}\, a(m, Q^*) + a(mq, Q) = (1+q^{11-l})\frac{65520}{691}\sigma_{11}(m) + a_2\, \tau(m).$$

[$a(m, Q)$ denotes the number of representations of m by the form Q].

IV. There exists a cusp form of dimension -2 and of grade 11.

$$j(\tau) = \sqrt[12]{\Delta(\tau)\cdot\Delta(11\tau)} = x - 2x^2 - x^3 + 2x^4 + x^5 + 2x^6 - 2x^7 + 0x^8 + \cdots$$

There exist three classes of quadratic forms of determinant 121 and grade 11 in 4 variables. (Hecke [5], pp. 116–117).

V. It follows from the congruence properties of $\tau(n)$ and from the well known fact that the functions $\sigma_k(n)$ for odd values of k are

divisible by every number r for almost all numbers n (e.g. Hardy [3] page 167)

that $\tau(n) \equiv 0 \pmod{2^{10} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 691}$ Lahiri [3]

for almost all numbers n; that is to say the number $N(x)$ of exceptional numbers n, which are less than x, satisfies $N(x) = o(x)$.

7. References.

This list is certainly incomplete concerning publications before 1940. We refer to other lists in *Collected papers of S. Ramanujan* C.U.P. 1927; Hardy [3]; Hardy Wright [6]; Kloosterman [2]. The first number refers to the paragraph of this paper, which is connected with the publication mentioned.

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THE INVARIANTS OF FINITE TRANSFORMATION GROUPS—I

BY

P. KESAVA MENON

1. The object of this paper is to study the invariants of finite groups of transformations in a purely algebraic manner.

Our coefficient domain shall be an arbitrary commutative field F, finite or infinite. By functions we shall mean rational functions of one or more variables with coefficients in F.

2. Let

$$G = \{ S_1, S_2, \cdots, S_N \}$$

be a finite group of transformations defined by

$$S_i\ (x) = \frac{a_i\ x + b_i}{c_i\ x + d_i},\ a_i,\ b_i,\ c_i,\ d_i,\ \varepsilon\ F. \quad (i = 1, 2, \cdots, N).$$

We may suppose that S_1 is the identity element of the group:

$$S_1\ (x) = x,$$

and hence that

$$a_1 = d_1 = 1,\ c_1 = b_1 = 0.$$

The function

$$A\ (t,\ x) = \prod_{i=1}^{N} \{t - S_i\ (x)\}$$

is obviously invariant under the transformations (S_i) applied to the variable x.

If P is an arbitrary transformation defined by

$$P\ (x) = \frac{\alpha x + \beta}{\gamma x + \delta}$$

so that its inverse P^{-1} is

$$P^{-1}\ (x) = \frac{\delta x - \beta}{-\gamma x + \alpha},$$

then

$$t - P^{-1}\ (x) = \left(\frac{\gamma t + \delta}{\gamma x - \alpha}\right) \{x - P\ (t)\}.$$

Letting P run through the transformations S_i and taking the product, we get

$$A\ (t,\ x) = \prod_{i=1}^{N} \{t - S_i^{-1}\ (x)\} = \prod_{i=1}^{N} \left(\frac{c_i\ t + d_i}{c_i\ x - a_i}\right) \prod_{i=1}^{N} \{x - S_i\ (t)\}$$

$$= \prod_{i=1}^{N} \left(\frac{c_i\ t + d_i}{c_i\ x - a_i}\right) A(x,\ t) \qquad (1)$$

so that

$$\frac{A(t, x)}{\prod_{i=1}^{N}(c_i t + d_i)} = \frac{A(x, t)}{\prod_{i=1}^{N}(c_i x - a_i)}. \tag{2}$$

Since the right-hand member in this relation is invariant when the transformations of G are applied to t and the left-hand member is invariant, as already remarked, when the transformations are applied to x, it follows that the function

$$B(t, x) = \frac{A(t, x)}{\prod_{i=1}^{N}(c_i t + d_i)}$$

is invariant when the transformations of G are applied to x or to t.

3. Let us consider the function

$$\varphi(t) = B(t, x) - B(t, y) = \frac{A(t, x) - A(t, y)}{\prod_{i=1}^{N}(c_i t + d_i)}.$$

This is a rational function of t of degree less than N since $A(t, x) - A(t, y)$ is of degree at most $N-1$ in t and $c_1 = 0$. Moreover $\varphi\{S_i(y)\} = B\{S_i(y), x\} - B\{S_i(y), y\} = B(y, x) - B(y, y) = B(y, x)$ so that the function

$$\varphi(t) - B(y, x)$$

vanishes for the N values $S_i(y)$ of t, and therefore is identically zero.

Thus we get

$$B(t, x) - B(t, y) = B(y, x).$$

Putting $t = x$ we get.

$$B(y, x) = -B(x, y) \tag{3}$$

so that the above relation becomes

$$B(t, x) + B(x, y) + B(y, t) = 0. \tag{4}$$

4. Let p be some element of the field F, extended if necessary, such that

$$c_i p + d_i \neq 0 \quad (i = 1, 2, \cdots, N),$$

and write

$$I(t) = B(t, p).$$

If q is any other element such that

$$c_i q + d_i \neq 0 \quad (i = 1, 2, \cdots, N)$$

then the relation

$$B(t, p) - B(t, q) = B(q, p)$$

shows that $I(t)$ is determinate up to an additive constant. We may now express $B(t, x)$ in terms of I:

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$$B(t, x) = B(t, p) - B(x, p) = I(t) - I(x).$$

5. We proceed to prove two important properties of the function $I(t)$.

Let P be any transformation which leaves $I(t)$ invariant. Then

$$B[P(x), x] = I[P(x)] - I(x) = 0,$$

so that

$$\prod_{i=1}^{N} \{P(x) - S_i(x)\} = 0.$$

Since the factors on the left side are elements of the field $F(x)$ of rational functions over F, their product cannot vanish without one of them being zero. Therefore

$$P(x) = S_i(x)$$

for some i, and so the only transformations which leave $I(t)$ invariant are those of the group G.

We next show that every function of x which is invariant under the transformations of G is a function of $I(x)$. The following proof is on the same lines as that of a special case discussed by E. Artin.[1]

The transformations S_i are automorphisms of the field $F(x)$ of all rational functions of x with coefficients in F. Let K be the fixed-point field under these automorphisms. Then $F(x)$ is an extension of K of degree at least N. Moreover the field K contains the field $F\{I(x)\}$ of all rational functions of $I(x)$ with coefficients in F. Hence our result will follow if we show that $F(x)$ is an extension of $F\{I(x)\}$ of degree at most N. For this we have only to show that x is the root of an equation of degree at most N with coefficients in $F\{I(x)\}$. In fact such an equation is given by $I(t) - I(x) = 0$, or

$$A(t, p) - I(x) \prod_{i=1}^{N} (c_i t + d_i) = 0.$$

6. From (3) and (2) we get

$$\frac{A(t, x)}{\prod_{i=1}^{N} (c_i t + d_i)} = -\frac{A(x, t)}{\prod_{i=1}^{N} (c_i x + d_i)} = -\frac{A(t, x)}{\prod_{i=1}^{N} (c_i t - a_i)},$$

so that

$$\prod_{i=1}^{N} (c_i t + d_i) = -\prod_{i=1}^{N} (c_i t - a_i). \tag{6}$$

Hence, also

$$A(t, x) - A(t, y) = \{I(x) - I(y)\} \prod_{i=1}^{N} (c_i t - a_i). \tag{7}$$

Let us write

$$\sum_{i=1}^{N} S_i(x) = P_1(x)$$

$$\sum_{i<j} S_i(x)\, S_j(x) = P_2(x)$$

$$S_1(x)\, S_2(x) \cdots S_N(x) = P_N(x);$$

and

$$\sum a_{i_1} a_{i_2} \cdots a_{i_r} c_{j_1} c_{j_2} \cdots c_{j_{N-r}} = k_r \quad (r = 1, 2, \cdots, N)$$

where the summation on the left-hand side is over all permutations $(i_1, i_2, \cdots, i_r, j_1, j_2, \cdots, j_{N-r})$ of the numbers 1, 2, $\cdots$, N. Then, on equating coefficients in (7) we get

$$P_r(x) - P_r(y) = k_r \{I_r(x) - I_r(y)\}$$

or

$$P_r(x) - k_r I_r(x) = P_r(y) - k_r I_r(y).$$

It follows that

$$P_r(x) = k_r I_r(x) + \text{constant} \quad (r = 1, 2, \cdots, N).$$

In particular if more than r of the c_i's or more than $N - r$ of the a_i's are zero, then $k_r = 0$ and P_r reduces to a constant.

In passing, we observe that from the equality

$$\prod_{i=1}^{N} S_i(t) = \prod_{i=1}^{N} S_i^{-1}(t)$$

we have

$$\frac{\prod_{i=1}^{N} (a_i t + b_i)}{\prod_{i=1}^{N} (c_i t + d_i)} = \frac{\prod_{i=1}^{N} (b_i - d_i t)}{\prod_{i=1}^{N} (c_i t - a_i)}$$

so that (6) gives

$$\prod_{i=1}^{N} (a_i t + b_i) = - \prod_{i=1}^{N} (b_i - d_i t).$$

7. The derivative $I'(x)$ of $I(x)$ can be written in an elegant form. In fact, putting $t = x$ in the relation

$$\frac{I(t) - I(x)}{t - x} = \frac{B(t, x)}{t - x} = \prod_{i=2}^{N} \left\{ \frac{t - S_i(x)}{c_i t + d_i} \right\}$$

we get

$$I'(x) = \prod_{i=2}^{N} \left\{ \frac{x - S_i(x)}{c_i x + d_i} \right\} = \prod_{i=2}^{N} \frac{c_i x^2 + (d_i - a_i)x - b_i}{(c_i x + d_i)^2}. \tag{8}$$

Using (6) we may also write

$$\frac{1}{I'(x)} = - \prod_{i=2}^{N} \frac{(c_i x + d_i)(c_i x - a_i)}{c_i x^2 + (d_i - a_i) x - b_i} = - \prod_{i=2}^{N} \left\{ c_i - \frac{(a_i d_i - b_i c_i)}{c_i x^2 + (d_i - a_i)x - b_i} \right\}.$$

Integrating (8) from x to t, we get back B (t, x):

$$B(t, x) = \int_x^t \prod_{i=2}^{N} \frac{c_i x^2 + (d_i - a_i)x - b_i}{(c_i x + d_i)^2} dx.$$

Differentiating the relation

$$I\{S_i(x)\} = I(x)$$

we get

$$I'\{S_i(x)\} \frac{a_i d_i - b_i c_i}{(c_i x + d_i)^2} = I'(x).$$

Taking the product from $i=1$ to N we have

$$\prod_{i=1}^{N} I'\{S_i(x)\} = \{I'(x)\}^N \prod_{i=1}^{N} \frac{(c_i x + d_i)^2}{(a_i d_i - b_i c_i)}.$$

Since the left-hand member is invariant under the transformations of G we get, on substituting for $I'(x)$ from (8), the function

$$\prod_{i=2}^{N} \frac{\{c_i x^2 + (d_i - a_i)x - b_i\}^N}{(c_i x + d_i)^{2N-2}}$$

as also invariant under the transformations of G.

8. Between any three polynomials A (t, x), A (t, y), A (t, z) in t, there exists a linear relation. This is evident from (7); but it may be more instructive to deduce it as follows.

The function

$$\psi(t, x, z) = \frac{A(t, x)}{A(t, z)} = \frac{B(t, x)}{B(t, z)}$$

is invariant when the transformations of G are applied to t, x or z. Let us seek a constant λ for which

$$\psi(t, x, z) - \lambda\psi(t, y, z) = \frac{A(t, x) - \lambda A(t, y)}{A(t, z)}$$

is of degree less than N in t. For this it is necessary and sufficient that the equations

$$A(t, x) - \lambda A(t, y) = 0$$
$$A(t, z) = 0$$

have a common root. But the roots of the second equation are

$$S_i(z) \quad (i = 1, 2, \cdots, N).$$

Putting $t = S_i(z)$ in the first equation we get

$$\lambda = \frac{A\{S_i(z), x\}}{A\{S_i(z), y\}} = \psi\{S_i(z), x, y\} = \psi(z, x, y).$$

It follows that for this value of λ the two equations have the same N roots. Thus we see that the function

$$\psi(t, x, z) - \psi(z, x, y)\,\psi(t, y, z)$$

independent of t. Putting $t = y$ we get

$$\psi(t, x, z) - \psi(z, x, y)\,\psi(t, y, z) = \psi(y, x, z). \tag{9}$$

Substituting the values of the ψ's, this becomes

$$A(t, x)\,A(y, z)\,A(z, y) = A(t, y)\,A(z, x)\,A(y, z) + A(t, z)\,A(y, x)\,A(z, y).$$

The relation (9), may, of course, be readily obtained by expressing ψ in terms of I:

$$\psi(t, x, z) = \frac{I(t) - I(x)}{I(t) - I(z)}.$$

9. Among the transformations of G, the identity S_1 has the property that $c_1 = 0$. Let us suppose that the set

$$H = \{S_1, S_2, \cdots, S_r\}$$

are the transformations of G such that

$$c_i = 0 \quad (i = 1, 2, \cdots, r)$$

and that

$$c_i \neq 0 \quad (i = r+1, \cdots, N).$$

Then it is obvious that H is a sub-group of G. Consider the cosets

$$S'H,\ S''H,\ \ldots,\ S^{(\mu-1)}H \quad (\mu r = N)$$

of H in G. Let $q', q'', \ldots, q^{(\mu-1)}$ respectively be the elements of the field F for which the denominators of $S'(x), S''(x), \ldots, S^{(\mu-1)}(x)$ vanish. Thus, for instance, if $S'(x) = \dfrac{ax+b}{cx+d}$, then $q' = -d/c$. It is then evident that when the transformations of the same coset are applied to x, the denominators all vanish for the same value of x. It follows that

$$\prod_{i=1}^{N} (c_i x + d_i) = (d_1 d_2 \ldots d_r c_{r+1} \ldots c_N)\{(x - q')(x - q'')\ldots(x - q^{(\mu-1)})\}^r.$$

If we suppose at the outset that $d_1 = \cdots = d_r = c_{r+1} = \cdots = c_N = 1$, as we may do without loss of generality, we may therefore write

$$B(t, x) = \frac{A(t, x)}{\{(x - q')(x - q'')\ldots(x - q^{(\mu-1)})\}^r}.$$

For certain values of x, belonging at most to some quadratic extension of the coefficient field F, the numerator of $B(t, x)$ also will reduce to powers of polynomials. Let p be a fixed point for some transformations of G other than the identity. These, along with the identity, form a sub-group K of G of order s, say. Let, further,

$$G = K + S'K + S''K + \cdots + S^{(\nu-1)}K \quad (\nu s = N)$$

be the resolution of G into cosets of K. Then if $S', S'', \ldots, S^{(\nu-1)}$

move p to $p', p'', \ldots, p^{(\nu-1)}$ respectively, we see that all the transformations of the coset $S^{(j)}K$ move p to $p^{(j)}$ and hence that

$$A(t, p) = \prod_{i=1}^{N} \{t - S_i(p)\} = \{(t-p)(t-p')\ldots(t-p^{(\nu-1)}\}^s .$$

Hence, if in the definition of $I(t)$ we take p to be the above quantity, we have

$$I(t) = \frac{\{(t-p)(t-p')\ldots(t-p^{(\nu-1)})\}^s}{\{(t-q')(t-q'')\ldots(t-q^{(\mu-1)})\}^r} .$$

10. Let us next consider the effect of transforming the group G by an arbitrary transformation P.

$$\text{If } P(x) = \frac{\alpha x + \beta}{\gamma x + \delta},$$

then from (1)

$$t - P^{-1}SP(x) = \left\{ \frac{\gamma t + \delta}{\alpha - \gamma SP(x)} \right\} \{P(t) - SP(x)\}.$$

Hence, writing

$$A^*(t, x) = \prod_{i=1}^{N} \{t - P^{-1}S_i P(x)\}$$

we get

$$\frac{A^*(t, x)}{(\gamma t + \delta)^N} = \frac{A[P(t), P(x)]}{\prod_{i=1}^{N} \{\alpha - \gamma S_i P(x)\}} .$$

If $\gamma \neq 0$, we may write this as

$$\frac{A^*(t, x)}{(t + \delta/\gamma)^N} = \frac{A[P(t), P(x)]}{A[\alpha/\gamma, P(x)]} .$$

and so

$$\frac{A^*(t, x)}{A^*(t, y)} = \frac{A[\alpha/\gamma, P(y)]}{A[\alpha/\gamma, P(x)]} \cdot \frac{A[P(t), P(x)]}{A[P(t), P(y)]} .$$

If I^* is the invariant of the group $G^* = P^{-1}GP$ corresponding to the invariant I of G, then

$$\frac{A^*(t, x)}{A^*(t, y)} = \frac{I^*(t) - I^*(x)}{I^*(t) - I^*(y)}$$

and so

$$\frac{I^*(t) - I^*(x)}{I^*(t) - I^*(y)} = \left\{ \frac{I(\alpha/\gamma) - P(y)}{I(\alpha/\gamma) - P(x)} \right\} \left\{ \frac{I[P(t)] - I[P(x)]}{I[P(t)] - I[P(y)]} \right\} .$$

Finally, solving for $I^*(t)$ we get

$$I^*(t) = \frac{a\, I[P(t)] + b}{c\, I[P(t)] + d}$$

where a, b, c, d are constants.

If on the other hand $\gamma = 0$, then

$$A^*(t, x) = \left(\frac{\delta}{\alpha}\right)^N A[P(t), P(x)]$$

so that

$$\frac{I^* (t) - I^* (x)}{I^* (t) - I^* (y)} = \frac{I [P (t)] - I [P (x)]}{I [P (t)] - I [P (y)]},$$

whence we again get

$$I^* (t) = \frac{a\, I [P (t)] + b}{c\, I [P (t)] + d}$$

where a, b, c, d are constants.

If in particular the group G^* is the same as the group G, we may take

$$I^* (t) = I (t)$$

so that

$$I [P (t)] = \frac{\alpha^* I (t) + \beta^*}{\gamma^* I (t) + \delta^*} = P^* [I (t)], \text{ say},$$

where α^*, β^*, γ^*, δ^* are constants.

Let us now suppose that G is a normal subgroup of some group K. Then resolving K into cosets of G we have

$$K = G + PG + QG + \cdots$$

where P, Q, . . . are certain transformations in K.

For any transformation PS_i belonging to PG we have

$$I [PS_i (t)] = I [S_i P (t)] = I [P (t)] = P^* [I (t)].$$

Similarly the transformations belonging to the coset QG correspond to the same transformation Q^* of $I (t)$ and so on. Moreover

$$I [PQ (t)] = P^* [I \{Q (t)\}] = P^* Q^* [I (t)]$$

so that P^*Q^* corresponds to PQG. It follows that the set of transformations

$$(I, P^*, Q^*, \ldots)$$

form a group which is isomorphic to the quotient group K/G. Thus we get a representation of the quotient group.

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A NOTE ON GÖTLIND'S AXIOM SYSTEM FOR THE CALCULUS OF PROPOSITIONS

BY

R. K. P. SINGH AND R. SHUKLA, *Patna*

Erik Götlind [1] has proposed a set of axioms for the calculus of propositions (of the first order). Unfortunately, we have no access to the original paper in Swedish. The reviewer says that the author has shown the system to be sufficient for the calculus. In fact the proposed system is a modification of the system of Hilbert and Ackermann.[2] The author, in his paper, raises the question of independence. This note proves the independence of the axioms by the usual matrix method.

The following is the system under investigation:

(*a*) $p \vee p \rightarrow p$

(*b*) $p \rightarrow p \vee q$

(*c*) $p \rightarrow p$

(*d*) $p \rightarrow r. \rightarrow : q \vee p. \rightarrow. \; r \vee q.$

In the following matrices 1 is the truth-value *false* (or the undesignated truth-value in the sense of P. Bernays); and 0, 2, and 3 (the last whenever it occurs) are the truth-values *True* (or designated).

1. For the independence of axiom (*a*), consider the following matrices defining V and $\rightarrow$:

V	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$\rightarrow$	0	1	2
0	0	1	2
1	0	0	1
2	0	0	0

Here (*a*) takes the truth-value 1 for $p=1$, and the rest of the axioms take the truth value 0 for all values of p, q and r.

2. For the axiom (*b*), consider the following matrices:

V	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\rightarrow$	0	1	2	3
0	0	1	1	1
1	0	0	0	0
2	0	1	2	3
3	0	1	2	2

Now, for $p=0$ or 1, (a) takes the truth value 0;
and for $p=2$ or 3, (a) takes the truth value 2.
But for $p=2, q=1$, $(b) .\equiv. 2 \rightarrow (2 \vee 1) .\equiv. 2 \rightarrow 1 = 1$.

Again for $p=0$ or 1, (c) takes the truth-value 0;
and for $p=2$ or 3, (c) takes the truth-value 2.

Again for $p=0$ or 1, any value of q and r, $(d)=0$
for $p=2$, $q=0$ or 1, any r, $(d)=0$
for $p=2$, $q=2$ or 3, $r=0$ or 1, $(d)=0$
for $p=2$, $q=2$ or 3, $r=2$ or 3, $(d)=2$
for $p=3$, $q=0$ or 1, any r, $(d)=0$
for $p=3$, $q=2$ or 3, $r=0$ or 1, $(d)=0$
for $p=3$, $q=2$ or 3, $r=2$ or 3, $(d)=2$.

3. To shew the independence of axiom (c), we consider the following matrices:

V	0	1	2
0	0	0	0
1	0	0	0
2	0	0	0

→	0	1	2
0	0	0	0
1	0	0	2
2	0	2	1

Here for all values of p, q and r, the axioms (a), (b) and (d) take the truth-value 0; but for $p=2$, the axiom (c) takes the truth-value 1.

4. For the independence of (d), we consider the following matrices:

V	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	1	2	3
3	0	0	3	2

→	0	1	2	3
0	0	1	2	3
1	0	0	3	2
2	0	0	0	0
3	0	0	0	0

It can be seen that for all values of p, q, r each of the axioms (a), (b) and (c) takes the truth value 0; but for $p=3$, $q=1$, $r=1$,

$$(d) \equiv [3 \rightarrow 1. \rightarrow : 1 \vee 3. \rightarrow. 1 \vee 1]$$
$$.\equiv. 0 \rightarrow. 0 \rightarrow 1$$
$$.\equiv. 0 \rightarrow 1$$
$$= 1,$$

This completes the proof of the independence of the system. The consistency proof is trivial—the usual Hilbert and Ackermann proof completely goes through. It is to be observed that the implication matrices are generally non-normal. It will be interesting to have the proof by normal matrices.

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ON THE UPPER BOUND OF $\sum_{i=1}^{n} x_i^m$ SUBJECT TO THE CONDITIONS $\sum x_i = 0$ and $\sum x_i^2 = n$

BY

M. LAKSHMANAMURTI, *Andhra University, Waltair*

Let $\alpha_m = \frac{1}{n} \sum_{i=1}^{n} x_i^m$ subject to the conditions

$$h \equiv x_1 + x_2 + \cdots + x_n = 0 \tag{1}$$

$$g \equiv x_1^2 + x_2^2 + \cdots + x_n^2 = n \tag{2}$$

where the x's are real:
α_3 and α_4 play an important part in skewness and kurtosis. It is of interest to determine the upper bound of α_m as a function of n on the bounded closed set of points x given by equations (1) and (2). Ernest Wilkins[1], established that $\alpha_3 \leqslant \frac{n-2}{(n-1)^{1/2}}$ and that this value is attained at the point P_1:

$$x_1 = (n-1)^{1/2},\ x_2 = x_3 = \cdots = x_n = -(n-1)^{-1/2}.$$

Chakrabarti[2], established the above result using Sturm's theorem and also showed that $\alpha_4 \leqslant n-2+(n-1)^{-1/2}$ and that this value is attained at the same point P_1. Both the authors established the well known result: $\alpha_4 \geqslant \alpha_3^2 + 1$ originally due to Karl Pearson. I have in this paper established that α_m attains its upper bound at the point P_1 when m is not necessarily a positive integer. I further consider the more general problem of extremum of α_m for certain rational values of m. I establish the more general inequality: $\alpha_{2m} \geqslant \alpha^2_{m+1} + \alpha^2_m$, and obtain the conditions under which this inequality reduces to an equality.

1. From (1), it is seen that some of the x's are negative numbers, so that α_m is real only if $m = \frac{p}{q}$ where p and q are positive integers in their lowest terms and q is odd. Further we limit ourselves to the case $m > 3$. The problem of upper bound is in essence the problem of finding the maximum of α_m, that is of

$$f \equiv \sum_{i=1}^{n} x_i^m \tag{3}$$

subject to (1) and (2). Employing Lagrange's method of multipliers, we construct the function $F \equiv f + \lambda g + \mu h$. At the maximizing

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point the x's must satisfy the equations

$$\frac{\partial f}{\partial x_i} \equiv m x_i^{m-1} + 2 \lambda x_i + \mu = 0 ; \quad i = 1, 2, \ldots, n. \tag{4}$$

Multiplying (4) by x_i and iterating i from 1 to n, we get,

$$m \alpha_m + 2 \lambda = 0. \tag{5}$$

Adding equations (4), we get,

$$m \alpha_{m-1} + \mu = 0. \tag{6}$$

Eliminating λ and μ between (4), (5) and (6) we find that the x's must satisfy:

$$t^{m-1} - \alpha_m t - \alpha_{m-1} = 0. \tag{7}$$

α_m and α_{m-1} themselves involve the x's, but whatever real numbers x's may be, $\alpha_{p/q}$ is real and is moreover positive when p is even.

We easily find that (7) cannot admit more than two real roots when p is odd and more than three real roots when p is even. Thus when p is odd at an extremum point, the x_is form a two-point set, and when p is even they may form a two-point set, or a three-point set. The two-point set is found by putting $x_1 = x_2 = \cdots = x_k = a$ and $x_{k+1} = \cdots = x_n = b$ in (1) and (2), and solving them. We get

$$a = \left(\frac{n-k}{k}\right)^{\frac{1}{2}}, \qquad b = -\left(\frac{k}{n-k}\right)^{\frac{1}{2}}.$$

We represent this by P_k, $k = 1, 2, \ldots, n-1$. It is easily verified, by forming α_m and α_{m-1} for this set, that $t = a$ and $t = b$ satisfy (7) whether p is even or odd.

2. We shall now determine three-point sets at which an extremum can occur. Let (1) and (2) be solved for x_1 and x_2 and substituted in (3), yielding a function of $n-2$ variables $x_3, \ldots, x_n$. Let x_0 and y_0 be two unequal numbers and $x_1 = x_0$, $x_2 = y_0$ and $x_i = x_0$, $i \neq 1$, or 2. At this point,

$$\frac{1}{2m} \frac{\partial^2 f}{\partial x_i^2} = (m-1) x_0^{m-2} - \frac{x_0^{m-1} - y_0^{m-1}}{x_0 - y_0} \tag{8}$$

> 0 if $m > 2$; $|x_0| \geqslant |y_0|$

< 0 if $m > 2$; x_0 and y_0 are of same sign and $|x_0| < |y_0|$. Throughout this section, we treat p as even. Consider a three-point set:

$x_1 = x_2 = \ldots = x_r = x$, $x_{r+1} = x_{r+2} = \ldots = x_{r+s} = y$, $x_{r+s+1} = \ldots = x_n = z$ where no two of x, y, z are equal, and $r > 1$, $s > 1$, $n - r - s > 1$. Let x and y be of the same sign and $|x| > |y|$. Solving (1) and (2) for x_r and x_{r+1} and substituting in (3) we find that, by (8)

$$\frac{\partial^2 f}{\partial x_i^2} = +^{ve} \qquad \text{if } i = 1, 2, \ldots, r-1$$
$$= -^{ve} \qquad \text{if } i = r+2, \ldots, r+s.$$

Therefore for a possible maximum the set must be of the form $x_1 = x$, $x_2 = y$ and $x_i = z$, $(i \neq 1, 2)$. Substituting these in (7) and eliminating α_m and α_{m-1}, we obtain :

$$\begin{vmatrix} x^{m-1} & y^{m-1} & z^{m-1} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Also $\qquad x + y + (n-2)\, z = 0.$

Eliminating z between these two and putting $x = ty$ we get :

$$\begin{vmatrix} \{(n-2)\, t\}^{m-1} & (n-2)^{m-1} & (1+t)^{m-1} \\ (n-2)\, t & n-2 & (1+t) \\ 1 & 1 & -1 \end{vmatrix} = 0. \tag{9}$$

(9) is a reciprocal equation, having the roots, 1, -1, $-(n-1)$, $-\dfrac{1}{n-1}$. These roots, except -1, lead to $x = y = z$ which must be rejected. The root -1 leads to $x = -y$ and $z = 0$, so that [using (2)] a possible extremum is at the point $\left(\sqrt{\frac{n}{2}},\ -\sqrt{\frac{n}{2}},\ 0, 0, \ldots, 0\right)$.

3. We now proceed to show that (9) has no other real roots. For this purpose we need explore the open intervals $(0, 1)$; $-1, -(n-1)$ and $-(n-1), -\infty$. We can write (9) in the form

$$t\,(1 - t^{m-2}) + \frac{1+t}{n-2}(1 - t^{m-1}) - (1-t)\left(\frac{1+t}{n-2}\right)^{m-1} = 0. \tag{10}$$

For $0 < t < 1$, the left side of (10)

$$> 0 + \left\{\frac{1+t}{n-2} - \left(\frac{1+t}{n-2}\right)^{m-1}\right\}(1-t);\ m > 2,\ n \geqslant 4$$
$$> 0.$$

Changing t into $-t$, and dividing by $(t-1)\left(\dfrac{t-1}{n-2} - 1\right)$, we can express (10) in the form

$$t\,\frac{t^{m-2}-1}{t-1} - \frac{1+t}{n-2}\cdot\frac{\left(\frac{t-1}{n-2}\right)^{m-2} - 1}{\left(\frac{t-1}{n-2}\right) - 1} = 0. \tag{11}$$

If $1 < t < n-1$, and $m > 3$, the left side of (11)

$$> (m-2)\, t - \frac{1+t}{n-2}\cdot(m-2) = \frac{m-2}{n-2}\left\{(n-3)\, t - 1\right\} > 0.$$

We now consider the range $t > n-1$. Put $\dfrac{t-1}{n-2} = x$, so that $> 2x$, for $n \geqslant 4$, $x > 1$.

Since $\frac{1+t}{n-2} = \left(\frac{t+1}{t-1}\right) x < 2x$, the left side of (11) is
$> t\, t^{m-3} - 2x\,(m-2)\; x^{m-3}$
$> \{2^{m-2} - 2\,(m-2)\}\; x^{m-2} > 0$ if $m \geqslant 4$.
But if we take $n \geqslant 5$, then $t > 3x$, the left side of (11) is now
$> \{3^{m-2} - 2\,(m-2)\}\; x^{m-2} > 0$ if $m > 3$.

4. We now examine which of the two-point sets P_k give a maximum or minimum or neither. Solving (1) and (2) for x_k and x_{k+1} and substituting in (3), we get a function of $n-2$ variables and by (8) we have,

$$\left.\begin{aligned} \frac{1}{2m}\frac{\partial^2 f}{\partial x_i^2} &= (m-1)\,a^{m-2} - \frac{a^{m-1}-b^{m-1}}{a-b} = \text{A}\,;\ i<k \\ &= (m-1)\,b^{m-2} - \frac{a^{m-1}-b^{m-1}}{a-b} = \text{B}\,;\ i>k+1 \end{aligned}\right\}; \tag{12}$$

on calculation, it will be found that,

$$\left.\begin{aligned} \frac{1}{2m}\frac{\partial^2 f}{\partial x_i \partial x_j} &= 0 \text{ if one of } i \text{ and } j \text{ is } < k \text{ and the other } > k+1 \\ &= \tfrac{1}{2}\,\text{A} \quad \text{if } i<k;\ j<k;\ i\neq j \\ &= \tfrac{1}{2}\,\text{B} \quad \text{if } i>k+1;\ j>k+1;\ i\neq j. \end{aligned}\right\} \tag{13}$$

If Q_k: $x_1 = a+h_1$, $x_2 = a+h_2$, $\cdot\cdot$, $x_{k-1} = a+h_{k-1}$, $x_{k+2} = -\frac{1}{a}+h_{k-2}$, $\cdot\cdot\cdot$ $x_n = -\frac{1}{a}+h_n$, (the value of f at Q_k) – (the value of f at P_k) is of constant sign for all variations of the h's according as the quadratic form:

$$\text{A}\,(h_1^2+h_2^2+\cdot\cdot+h^2_{k-1}) + \text{B}\,(h^2_{k+2}+\cdot\cdot+h^2_n) + \text{A}\sum_{\substack{i<k\\ j<k}} h_i h_j + \text{B}\sum_{\substack{i>k+1\\ j>k+1}} h_i h_j,\ (i\neq j), \tag{14}$$

is of constant sign or not. If (14) is definite negative, P_k gives a maximum, and if definite positive, P_k gives a minimum, otherwise P gives neither a maximum nor a minimum.

Let $m>2$, and p odd. Since $b = -\frac{1}{a}$, and $a = \left(\frac{n-k}{k}\right)^{1/2}$ we find that $\text{A} = \left(m-2+\frac{k}{n}\right)a^{m-2} + \frac{k}{m}\cdot\frac{1}{a^{m-2}} > 0$ and B = —ve.
It is easily verified that P_1 gives a maximum and P_{n-1} gives a minimum and the rest of the points give neither maximum nor minimum.

5. Let p be even, $m>2$. In this case if P_k is an extremum of some kind, P_{n-k} is an extremum of the same kind. So we restrict k to be $\leqslant \left[\frac{n}{2}\right]$, the greatest integer in $\frac{n}{2}$. Then $\text{A} > (m-2)\,a^{m-2} > 0$. We now examine for what values of a, B is positive or negative.

B is positive if $(m-1)\frac{1}{a^{m-2}}\left(a+\frac{1}{a}\right)-a^{m-1}-\frac{1}{a^{m-1}}$ is positive i. e. if $(m-1)a^2+(m-2)-(a^2)^{m-1}$ is positive. $\varphi(t)\equiv(m-2)+(m-1)t-t^{m-1}=0$ has just one positive root for $m>2$ and this root is <3. In fact as m increases the root approaches 1. If this root is a_0^2 $(a_0>1)$,

for $2<m<4$,	$2<a_0^2<3$
for $m=4$,	$a_0^2=2$
for $m>4$,	$1<a_0^2<2$.

At P_1, $a^2=n-1\geqslant 3$ for $n\geqslant 4$, so that B is negative and the corresponding quadratic form is negative. So P_1 and P_{n-1} give maxima. If $n=3$ and $2<m<3$; B is positive and so P_1 and P_2 give minima.

Let $n\geqslant 4$ and $m>2$. Corresponding to a given m, calculate approximately a_0^2. Find the least positive integer n_0 for which $\frac{n-n_0}{n_0}<a_0^2$. Then P_k gives a minimum for $k=n_0, n_0+1, \cdots, n-n_0$. P_k gives neither max. nor min. for other values of k.

EXAMPLE 1: Let $m=4$, $n=6$. Here $a_0^2=2$. P_1 and P_5 give maxima, and P_2, P_3, P_4 give minima. For, at P_2, $a=\sqrt{2}$; $\therefore$ A is + ve and $B=0$ and the form (14) now becomes simply Ah_1^2. At P_3, $a=1$ $\therefore$ A is + ve and B = + ve. The form (14) is now $A(h_1^2+h_2^2)+B(h_5^2+h_6^2)+A\,h_1h_2+B\,h_5h_6$:

EXAMPLE 2: $m=10$, $n=40$. P_1 and P_{39} give maxima. $P_{17}, \cdots, P_{23}$ give minima and the rest give neither maxima nor minima. Here $a_0^2=1{\cdot}4$ nearly. At P_{16}, $a=\sqrt{\frac{3}{2}}$ so that A is + ve and B is − ve, and the point gives neither maximum nor minimum. At P_{17}, $a=\sqrt{\frac{23}{17}}=\sqrt{1{\cdot}35\cdots}$, so that A and B are both + ve. The quadratic form (14) is then

$$=A\left(\sum_{i=1}^{16}h_i^2+\sum_{i\neq j}h_ih_j\right)+B\left(\sum_{i=19}^{40}h_i^2+\sum_{i\neq j}h_ih_j\right);$$

the sum of two definite positive quadratic forms and so P_{17} gives a minimum.

The only three-point set $\left(\sqrt{\frac{n}{2}}, -\sqrt{\frac{n}{2}}, 0, \cdots, 0\right)$ which can be a possible maximum turns out to be neither maximum nor minimum. For, $\frac{\partial^2 f}{\partial x_i^2}=\frac{1}{2}(m-4)a^{m-2}$ and $\frac{\partial^2 f}{\partial x_i\partial x_j}=\frac{1}{2}(m-2)a^{m-2}$ where $i>2$; $a=\sqrt{\frac{n}{2}}$

It follows then, that P_1 (and P_{n-1}) is the only point which gives a maximum. If the points x defined by (1) and (2) form a closed set,

the function α_m attains its upper bound at its maximum, at P_1 and the maximum is given by

$$n\,\alpha_m = \frac{(n-1)^{m-1} + (-1)^m}{(n-1)^{\frac{m}{2}-1}}.$$

6. Lastly we consider the quadratic form in A, B, C:

$$\frac{1}{n}\sum_{i=1}^{n} (A\,x_i^m + B\,x_i + b)^2 \equiv A^2\,\alpha_{2m} + B^2\,\alpha_2 + C^2$$

$$+ 2BC\,\alpha_1 + 2CA\,\alpha_m + 2AB\,\alpha_{m+1};\ n \geqslant 3. \qquad (15)$$

The necessary and sufficient condition for the quadratic form to be definite is

$$\begin{vmatrix} \alpha_{2m} & \alpha_{m+1} & \alpha_m \\ \alpha_{m+1} & \alpha_2 & \alpha_1 \\ \alpha_m & \alpha_1 & 1 \end{vmatrix} \geqslant 0 \qquad (16)$$

i.e. $\alpha_{2m} \geqslant \alpha^2_{m+1} + \alpha^2_m$, since $\alpha_2 = 1$, $\alpha_1 = 0$. The inequality reduces to an equality in the following cases:

(1) when $x_1 = x_2 = \cdots = x_k = \left(\frac{n-k}{k}\right)^{1/2}$; $x_{k+1} = \cdots = x_n = -\left(\frac{k}{n-k}\right)^{1/2}$, $k = 1, 2, \cdots, n-1$.

(2) Let m be an odd positive integer. (16) reduces to an equality if each term $A\,x_i^m + B\,x_i + C$ on the left side of (15) is zero. Let n_1 of the x_i's take the value x, n_2 of them take the value y and the remaining n_3, the value z, $(n = n_1 + n_2 + n_3)$ where x, y, z are the three real roots of the equation in t:

$$A\,t^m + B\,t + C = 0.$$

x, y, z must satisfy the relation:

$$x^m(y-z) + y^m(z-x) + z^m(x-y) = 0,\ x \neq y,\ y \neq z,\ z \neq x. \qquad (17)$$

(17) represents a cone with vertex at (0, 0, 0). x, y, z are further connected by the relations

$$n_1 x + n_2 y + n_3 z = 0 \qquad (18)$$

$$n_1 x^2 + n_2 y^2 + n_3 z^2 = n. \qquad (19)$$

The plane (18) through the origin cuts the cone (17) in a finite number of generators and these generators cut the ellipsoid (19) in a finite number of points. Also there are only a finite number of partitions of n into n_1, n_2 and n_3. At all such three point sets, (16) reduces to an equality.

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AN EXTENSION OF DEHN'S THEOREM ON THE APPROXIMATION OF A FUNCTION BY A POWER SERIES

BY

CHIA-SHUN YIH

University of British Columbia, Vancouver, Canada

1. In 1947, Max Dehn published a theorem* giving the necessary and sufficient condition for the existence of the approximation of a function of a single variable by a polynomial. In the present note the theorem is extended to apply to functions of two variables. The extension to functions of n variables is then obvious.

It may be noted that Taylor's theorem gives the existence and continuity of partial derivatives of total order n as a sufficient condition for the existence of the approximation

$$f(a+h,\, b+k) = f(a,b) + a_{1,0}h + a_{0,1}k + a_{2,0}h^2 + a_{1,1}hk + a_{0,2}k^2 + \cdots + \sum_{l+m=n} h^l k^m \left[a_{l,m} + \varepsilon_{l,m}(h,k)\right] \qquad (1)$$

with $\lim\limits_{\substack{h\to 0\\ k\to 0}} \varepsilon_{l,m}(h,k) = 0$, where l and m are two integers whose sum is n.

That this condition is not at all necessary justifies the present study. For further discussions in this respect, the reader is referred to Dehn's paper cited above.

2. We define

$$R_{1,0}(h,k) = \frac{f(a+h,\, b+k) - f(a,\, b+k)}{h}$$

$$R_{0,1}(h,k) = \frac{f(a+h,\, b+k) - f(a+h,\, b)}{k}$$

$$R_{1,1}(h,k) = \frac{R_{1,0}(h,k) - R_{1,0}(h,0)}{k} = \frac{R_{0,1}(h,k) - R_{0,1}(0,k)}{h}$$

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* "On the Approximation of a Function by a Power Series", *The Mathematics Student*, Vol. XV, 1947, pp. 79–82. The theorem is the following:

The existence of $S_k = \lim R_k(h)$ for $k = 1, 2, \cdots, n$, where

$$R_1(h) = \frac{f(a+h) - f(a)}{h}, \qquad R_k(h) = \frac{R_{k-1}(2h) - R_{k-1}(h)}{h}$$

is necessary and sufficient for the existence of the approximation:

$$f(a+h) = f(a) + ha_1 + h^2 a_2 + \cdots + h^n (a_n + \varepsilon(h))$$

with $\lim\limits_{h\to 0} \varepsilon(h) = 0$, $a_1 = S_1$, and $a_k = \dfrac{S_k}{1\cdot 3 \cdots (2^{k-1}-1)}$ for $k \geq 2$.

and, for l and m greater than 1,

$$R_{l,0}(h,k) = \frac{R_{l-1,0}(2h,k) - R_{l-1,0}(h,k)}{h}$$

$$R_{0,m}(h,k) = \frac{R_{0,m-1}(h,2k) - R_{0,m-1}(h,k)}{k}$$

$$R_{l,1}(h,k) = \frac{R_{l,0}(h,k) - R_{l,0}(h,0)}{k} = \frac{R_{l-1,0}(2h,k) - R_{l-1,0}(h,k)}{h}$$

$$R_{l,m}(h,k) = \frac{R_{0,m}(h,k) - R_{0,m}(0,k)}{h} = \frac{R_{l,m-1}(h,2k) - R_{l,m-1}(h,k)}{k}$$

$$R_{l,m}(h,k) = \frac{R_{l-1,m}(2h,k) - R_{l-1,m}(h,k)}{h} = \frac{R_{l,m-1}(h,2k) - R_{l,m-1}(h,k)}{k}.$$

Then we shall prove that the existence of

$$S_{l,m} = \lim_{\substack{h \to 0 \\ k \to 0}} R_{l,m}(h,k)$$

for all positive integral values of l and m including zero* such that $l+m \leqslant n$ is necessary and sufficient for the existence of approximation (1).

3. Consistency of the definition of $R_{,m}(h,k)$. Since the quantity $R_{l,m}(h,k)$ can be reached according to the definition given in § 2 in $\frac{(l+m)!}{l!\,m!}$ ways, it is necessary to prove the consistency of the definition It is obvious that $R_{o,p}(h,k)$ and $R_{p,o}(h,k)$ where p is any positive integer are well-defined, and a simple calculation will show the well-definedness of $R_{1,1}(h,k)$. To show that $R_{l,m}(h,k)$, which does not belong to any of the categories $R_{o,p}(h,k)$, $R_{p,o}(h,k)$, and $R_{1,1}(h,k)$, is well-defined, it must be noted that due to the special role that the subscript 1 plays in the definition of $R_{l,m}(h,k)$, separate treatment of the quantities $R_{l,1}(h,k)$ and $R_{l,m}(h,k)$, where l and m are greater than 1, is necessary.

Since $R_{o,p}(h,k)$ is uniquely determined, and since

$$R_{l,m}(h\ k) = \frac{R_{o,m}(h,k) - R_{o,m}(0,k)}{h}$$

$$= \frac{R_{o,m-1}(h,2k) - R_{o,m-1}(h,k) - R_{o,m-1}(0,2k) - R_{o,m-1}(0,k)}{hk}$$

* Since we did not define $R_{0,0}(h,k)$, the case when both l and m are zero is excluded.

gives the same result as

$$R_{l,m}(h,k) = \frac{R_{l,m-1}(h,2k) - R_{l,m-1}(h,k)}{k}$$
$$= \frac{R_{o,m-1}(h,2k) - R_{o,m-1}(0,2k) - R_{o,m-1}(h,k) - R_{o,m-1}(0,k)}{hk}$$

the well-definedness of $R_{l,m-1}(h,k)$ implies that of $R_{l,m}(h,k)$. This induction step, together with well-definedness of $R_{l,1}(h,k)$, establishes the consistency of the definition of $R_{l,m}(h,k)$. Similarly, $R_{l,1}(h,k)$ is well-defined.

We now proceed to prove by induction that $R_{l,m}(h,k)$, where l and m are both greater than 1, is well-defined. Assuming that $R_{l-1,m-1}(h,k)$, $R_{l,m-1}(h,k)$ and $R_{l-1,m}(h,k)$ are uniquely determined, we have, according to one way of construction,

$$R_{l,m}(h,k) = \frac{R_{l-1,m}(2h,k) - R_{l-1,m}(h,k)}{h}$$
$$= \frac{R_{l-1,m-1}(2h,2k) - R_{l-1,m-1}(2h,k) - R_{l-1,m-1}(h,2k) + R_{l-1,m-1}(h,k)}{hk}$$

and, according to the other,

$$R_{l,m}(h,k) = \frac{R_{l,m-1}(h,2k) - R_{l,m-1}(h,k)}{k}$$
$$= \frac{R_{l-1,m-1}(2h,2k) - R_{l-1,m-1}(h,2k) - R_{l-1,m-1}(2h,k) + R_{l-1,m-1}(h,k)}{hk}.$$

Since the two ways give the same result, this induction step together with the well-definedness of $R_{1,1}(h,k)$, $R_{1,m}(h,k)$ $(m > 1)$, and $R_{l,1}(h,k)$ $(l > 1)$ establishes the well-definedness of $R_{l,m}(h,k)$, where l and m are greater than 1.

4. Proof of necessity. From the existence of the approximation (1), it follows that

$$R_{l,o}(h,k) = a_{1,o} + a_{2,o}\,h + a_{1,1}\,k + a_{3,o}\,h^2 + a_{2,1}\,hk + a_{1,2}\,k^2 + \cdots + \sum_{\substack{l+m=n\\ l\neq 0}} h^{l-1}k^m[a_{l,m} + \varepsilon_{l,m}(h,k)]$$

with $\lim\limits_{\substack{h\to 0\\ k\to 0}} \varepsilon_{l,m}(h,k) = 0$. Hence $S_{1,o} = a_{1,o}$. Similarly, $S_{o,1} = a_{o,1}$, and $S_{1,1} = a_{1,1}$.

If now for one pair of indices l and m with $l+m < n$,

$$R_{l,m}(h,k) = \sum_{p=0}^{n-(l+m)} \sum_{q=0}^{n-(l+m)-p} (2^{p+1}-1)(2^{p+2}-1)\cdots \times (2^{p+l-1}-1)(2^{q+1}-1)(2^{q+2}-1)\cdots$$
$$\times (2^{q+m-1}-1)a_{l+p+m+q}\,h^p k^q + \sum_{r=0}^{n-(l+m)} h^{n-(l+m)-r} k^r \varepsilon_r(h,k) \qquad (4{\cdot}1)$$

with $\lim\limits_{\substack{h\to 0\\k\to 0}} \varepsilon_r(h,k)=0$, where the $l-1$ and $m-1$ numerical factors are to be omitted when $l=0$ or 1 and $m=0$ or 1, respectively, then the definition of $R_{l+1,m}(h,k)$ gives

$$R_{l+1,m}(h,k) = \sum_{p=1}^{n-(l+m)} \sum_{q=0}^{n-(l+m)-p} (2^p-1)(2^{p+1}-1)\cdots(2^{p+l-1}-1)(2^{q+1}-1)\cdots$$

$$\times(2^{q+m-1}-1)\,a_{l+p,\,m+q}h^{p-1}k^q + \sum_{r=0}^{n-(l+m+1)} h^{n-(l+m+1)-r}k^r\, t'_r(h,k)$$

$$= \sum_{p=0}^{n-l+m+1)} \sum_{q=0}^{n-(l+m+1)-p} (2^{p+1}-1)\cdots(2^{p+l}-1)(2^{q+1}-1)\cdots$$

$$(2^{q+m-1}-1)\,a_{(l+1)+p,\,m+q}h^p k^q + \sum_{r=0}^{n-(l+m+1)} h^{n-(l+m+1)-r}k^r\, t'_r(h,k)$$

with $\lim\limits_{\substack{h\to 0\\k\to 0}} t'_r(h,k)=0$. Consequently, (4·1) holds for $R_{l+1,m}(h,k)$. Similarly, (4·1) holds for $R_{l,m+1}(h,k)$. Since it holds for $R_{1,0}(h,k)$, $R_{0,1}(h,k)$, and $R_{1,1}(h,k)$, it holds in general for $R_{l,m}(h,k)$ where $l+m\leqslant n$, and we conclude that, with the same provision for the case when l or m equals 0 or 1,

$$S_{l,m}=\lim_{\substack{h\to 0\\k\to 0}} R_{l,m}(h,k)=1\cdot3\cdot7\cdots(2^{l-1}-1)\cdot1\cdot3\cdot7\cdots(2^{m-1}-1)\,a_{l,m} \tag{4·2}$$

so that $S_{l,m}$ exists where $l+m\leqslant n$.

5. Proof of Sufficiency. In the following $\varepsilon(h,k)$ will denote a quantity such that $\lim\limits_{\substack{h\to 0\\k\to 0}} \varepsilon(h,k)=0$, but for convenience it will not always denote the same quantity. Distinction will be made only when the final result has been obtained.

If $S_{p,q}$ exists where $p+q\leqslant n$, we have, letting $l+m=n$,

$$\begin{aligned} R_{l-1,m}(2h,2k) &= R_{l-1,m}(h,2k)+hS_{l,m}+h\,\varepsilon(h,k)\\ &= R_{l-1,m}(h,k)+h\,S_{l,m}+k\,S_{l-1,m+1}+h\,\varepsilon(h,k)+k\,\varepsilon(h,k)\\ &= R_{l-1,m}\left(\frac{h}{2},\frac{k}{2}\right)+h\left(1+\frac{1}{2}\right)S_{l,m}+k\left(1+\frac{1}{2}\right)S_{l-1,m+1}\\ &\quad + h\left[\varepsilon(h,k)+\frac{\varepsilon\left(\frac{h}{2},k\right)}{2}\right]+k\left[\varepsilon(h,k)+\frac{\varepsilon\left(h,\frac{k}{2}\right)}{2}\right]\\ &= \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ &= R_{l-1,m}\left(\frac{h}{2^r},\frac{k}{2^r}\right)+h\left(1+\frac{1}{2}+\cdots+\frac{1}{2^r}\right)S_{l,m}\\ &\quad +k\left(1+\frac{1}{2}+\cdots+\frac{1}{2^r}\right)S_{l-1,m+1}+h\left[\varepsilon(h,k)+\frac{\varepsilon\left(\frac{h}{2},k\right)}{2}\right. \end{aligned}$$

$$+\frac{\varepsilon\left(\frac{h}{4}, k\right)}{4}+\cdots+\frac{\varepsilon\left(\frac{h}{2^r}, k\right)}{2^r}\Bigg]+k\left[\varepsilon(h, k)+\frac{\varepsilon\left(h, \frac{k}{2}\right)}{2}+\cdots\cdots+\frac{\varepsilon\left(h, \frac{k}{2^r}\right)}{2^r}\right] \qquad (5\cdot1)$$

When $r \to \infty$, $R_{l-1, m}\left(\frac{h}{2^r}, \frac{k}{2^r}\right) \to S_{l-1, m}$. So

$$R_{l-1, m}(2h, 2k)=S_{l-1, m}+2h\, S_{l, m}+2k\, S_{l-1, m+1}+2h\varepsilon(h, k)+2k\varepsilon(h, k). \qquad (5\cdot2)$$

Similarly,

$$R_{l, m-1}(2h, 2k)=S_{l, m-1}+2h\, S_{l+1, m-1}+2k\, S_{l, m}+2h\, \varepsilon(h, k)+2k\, \varepsilon(h, k). \qquad (5\cdot3)$$

Since

$$\begin{aligned} R_{l-1, m-1}(4h, 4k) &= R_{l-1, m-1}(2h, 4k)+2h\, R_{l, m-1}(2h, 4k) \\ &= R_{l-1, m-1}(2h, 2k)+2k\, R_{l-1, m}(2h, 2k)+2h\, R_{l, m-1}(2h, 4k) \end{aligned}$$

we have, by virtue of (5·2) and (5·3),

$$\begin{aligned} R_{l-1, m-1}(4h, 4k) = {} & R_{l-1, m-1}(2h, 2k)+2h\, S_{l, m-1}+2k\, S_{l-1, m}+12hk\, S_{l, m} \\ & +4h^2\, S_{l+1, m-1}+4k^2\, S_{l-1, m+1}+4h^2\varepsilon(h, k)+4hk\, \varepsilon(h, k)+4k^2\, \varepsilon(h, k). \end{aligned}$$

Then following the procedure of (5·1),

$$\begin{aligned} R_{l-1, m-1}(4h, 4k) = {} & S_{l-1, m-1}+4h\, S_{l, m-1}+4k\, S_{l-1, m}+\frac{(4h)^2}{3}\, S_{l+1, m-1} \\ & +(4h)(4k)S_{l, m}+\frac{(4k)^2}{3}\, S_{l-1, m+1}+(4h)^2\, \varepsilon(h, k)+(4h)(4k)\, \varepsilon(h, k)+(4k)^2\varepsilon(h, k). \end{aligned} \qquad (5\cdot4)$$

This process can continue until one of the subscripts $l-1$ and $m-1$ becomes zero, when the definition of $R_{l, m-1}$ or $R_{l-1, m}$ are slightly different according to § 2. In order to actually carry out the approximation formula, let $n=3$, $l=2$, and $m=1$. From (5·4),

$$R_{1, 1}(2h, 4k)=S_{1, 1}+2h\, S_{2, 1}+4k\, S_{1, 2}+2h\, \varepsilon(h, k)+4k\, \varepsilon(h, k)$$

Also, in consistency with the definition of $R_{2, 0}$,

$$\begin{aligned} R_{2, 0}(2h, 4k) &= R_{2, 0}(2h, 0)+4k\, S_{2, 1}+k\varepsilon(h, k) \\ &= R_{2, 0}(h, 0)+h\, S_{3, 0}+4k\, S_{2, 1}+h\, \varepsilon(h, 0)+k\, \varepsilon(h, k) \\ &= R_{2, 0}\left(\frac{h}{2}, 0\right)+h\left(1+\frac{1}{2}\right) S_{3, 0}+4k\, S_{2, 0}+h\left[\varepsilon(h, 0)+\frac{\varepsilon\left(\frac{h}{2}, 0\right)}{2}\right] \\ &\quad +k\, \varepsilon(h, k) \\ &= \cdots\cdots\cdots \\ &= S_{2, 0}+2h\, S_{3, 0}+4k\, S_{2, 0}+2h\, \varepsilon(h, k)+4k\, \varepsilon(h, k). \end{aligned}$$

Hence, by substitution and iteration,

$$\begin{aligned} R_{1, 0}(4h, 4k) &= R_{1, 0}(2h, 4k)+2h\, R_{2, 0}(2h, 4k) \\ &= R_{1, 0}(2h, 0)+4k\, R_{1, 1}(2h, 4k)+2h\, R_{2, 0}(2h, 4k) \\ &= R_{1, 0}(2h, 0)+4k[S_{1, 1}+2h\, S_{2, 1}+4k\, S_{1, 2}+2h\, \varepsilon(h, k)+4k\, \varepsilon(h, k)] \\ &\qquad +2h\, [S_{2, 0}+2h\, S_{3, 0}+4k\, S_{2, 0}+2h\, \varepsilon(h, k)+4k\, \varepsilon(h, k)] \end{aligned}$$

$$= S_{1,0} + 4k\, S_{1,1} + 4h\, S_{2,0} + \frac{(4h)^2}{3} S_{3,0} + (4h)(4k) S_{2,1} + (4k)^2 S_{1,2}$$
$$+ (4h)^2 \varepsilon(h, k) + (4h)(4k)\, \varepsilon(h, k) + (4k)^2 \varepsilon(h, k)$$

or

$$R_{1,0}(h, k) = S_{1,0} + hS_{2,0} + kS_{1,1} + \frac{h^2}{3} S_{3,0} + hk\, S_{2,1} + k^2 S_{1,2} + h^2 \varepsilon(h, k)$$
$$+ hk\, \varepsilon(h, k) + k^2\, \varepsilon(h, k). \tag{5·5}$$

Since $R_{1,0}(h, k) = \dfrac{f(a+h, b+k) - f(a, b+k)}{h}$

and by Dehn's Theorem

$$f(a, b+k) = f(a, b) + k\, S_{0,1} + k^2 S_{0,2} + \frac{k^3}{3} S_{0,3} + k^3 \varepsilon(k),$$

we have

$$f(a+h, b+k) = f(a, b) + h\, S_{1,0} + k\, S_{0,1} + h^2 S_{2,0} + hk\, S_{1,1} + k^2 S_{0,2}$$
$$+ \frac{h^3}{3} S_{3,0} + h^2 k\, S_{2,1} + hk^2 S_{1,2} + \frac{k^3}{3} S_{0,3} + h^3 \varepsilon_0(h, k)$$
$$+ h^2 k\, \varepsilon_1(h, k) + hk^2\, \varepsilon_2(h, k) + k^3\, \varepsilon_3(h, k). \tag{5·6}$$

In general

$$f(a+h, b+k) = f(a, b) + \sum_{1 \leq l+m \leq n} \frac{S_{l,m}\, h^l k^m}{1·3·7\cdots(2^{l-1}-1)·1·3·7\cdots(2^{m-1}-1}$$
$$+ \sum_{r=0}^{n} h^{n-r} k^r\, \varepsilon_r(h, k) \tag{5·7}$$

where the $l-1$ and $m-1$ numerical factors in the denominator are to be omitted when $l=0$ or 1 and $m=0$ or 1, respectively, and where $\lim_{\substack{h \to 0 \\ k \to 0}} \varepsilon_r(h, k) = 0$ for $r = 1, 2, \ldots, n$.

6. *Remark.* As has been remarked by Dehn, a disadvantage of using our limits S consists in the arbitrariness connected with the comparison of the R's for $2h$ and h, or for $2k$ and k. We could as well have taken $rh\,(r > 1)$ instead of $2h$. The existence of the corresponding limits is equivalent to the existence of our limits S.

NOTES AND DISCUSSIONS

Reversible Number Sets with equal sums of like powers

BY

R. VENKATACHALAM IYER, *Trivandrum*.

At the last Mathematical Conference held at Madras, Mr. D. R. Kaprekar gave a variation of Tarry's problem in which the digits of the numbers on each side written in succession are reverses of each other. Thus,

$$4,\ 20,\ 42,\ 3,\ 41,\ 25 \overset{3}{=} 5,\ 21,\ 43,\ 2,\ 40,\ 24.$$

Supplying the figure zero before the elements 4, 3, 5 and 2 to make the number of digits in all elements equal, the problem may be regarded as a case of "Cyclic reversibility". Thus,

$$04,\ 20,\ 42,\ 03,\ 41,\ 25 \overset{3}{=} 05,\ 21,\ 43,\ 02,\ 40,\ 24.$$

This gives rise to the existence of two other types of equalities, like,

$$13,\ 42,\ 53,\ 57,\ 68,\ 97 \overset{3}{=} 79,\ 86,\ 75,\ 35,\ 24,\ 31$$

and $13,\ 56,\ 42,\ 46,\ 75,\ 43,\ 45,\ 32 \overset{3}{=} 35,\ 64,\ 24,\ 67,\ 54,\ 34,\ 53,\ 21.$

In the first case, the digits of the elements on one side when reversed give the elements on the other side, or each side is a mirror-reflection of the other side. In the second case, a cyclic change in the digits, in the order they are written on one side gives the elements on the other.

According to Tarry's lemma, the equality

$$a_1,\ a_2,\ \ldots,\ a_k \overset{n}{=} b_1,\ b_2,\ \ldots,\ b_k \text{ implies}$$

$$a_1,\ a_2,\ \ldots,\ a_k,\ b_1+d,\ \ldots,\ b_k+d \overset{n+1}{=} b_1,\ b_2,\ \ldots,\ b_k,\ a_1+d,\ a_2+d,\ \ldots,\ a_k+d.$$

Simple rules are given below for the derivation of the three types of equalities, starting with an equality of the first degree in each type and using Tarry's lemma.

Cyclic reversibility: Type 1.

Start with the equality $(10x+y)+(10z+w)=(10x+w)+(10z+y)$, where $x+z=y+w$ and $x,y,z,w < 9$. Apply Tarry's lemma with $d=1$ in succession (or a convenient multiple of 1 to avoid repetition of elements).

Example :—Let $x=1,\ y=3,\ z=4,\ w=2$.

We have, $13+42=12+43$.

$\therefore\ \ 13,\ 42,\ 14,\ 45 \overset{2}{=} 15,\ 44,\ 12,\ 43$, applying the lemma with $d=2$.

Applying the lemma to this equality with $d=3$, we get,

$$13, 42, 14, 45, 18, 47, 15, 46 \stackrel{3}{=} 16, 45, 17, 48, 15, 44, 12, 43.$$

This equality is of the type, as it stands. The terms 45, 15 can be cancelled from both sides, when it can be rearranged as follows.

$$13, 47, 18, 42, 14, 46 \stackrel{3}{=} 16, 44, 12, 48, 17, 43.$$

If in applying the lemma, $d=1$ had been used, instead of $d=3$, the resulting third degree equality would have been

$$13, 42, 14, 45, 16, 45, 13, 44 \stackrel{3}{=} 14, 43, 15, 46, 15, 44, 12, 43,$$

which is of the proper form. If 14 and 44 are cancelled from both sides, the equality can be rearranged as follows:

$$13, 42, 16, 45, 13, 45 \stackrel{3}{=} 15, 43, 15, 46, 12, 43.$$

Recognising the repetition of the elements 13, 45 on the left side and 15, 43 on the right side, the equality may be rearranged as

$$(13)_2, 42, 16, (45)_2 \stackrel{3}{=} (15)_2, 46, 12, (43)_2$$

or $$42, (13)_2, (45)_2, 16 \stackrel{3}{=} 46, (15)_2, (43)_2, 12.$$

Elements with digits reversed: Type 2.

Start with the equality

$$(10x+y)+(10z+w)=(10w+z)+(10y+x),$$

where $x+z=y+w$ and $x,y,z,w < 9$.

Apply Tarry's lemma with $d=11$ in succession (or a convenient multiple of 11 to avoid repetition of elements).

Example :—Let $x=1$, $y=3$, $z=4$, $w=2$.

we have, $13+42=24+31$.

$\therefore$ $13, 42, 46, 53 \stackrel{2}{=} 35, 64, 24, 31$ [taking $d=22$]

$\therefore$ $13, 42, 46, 53, 68, 97, 57, 64 \stackrel{3}{=} 46, 75, 79, 86, 35, 64, 24, 31.$

i.e., $13, 42, 53, 68, 97, 57 \stackrel{3}{=} 75, 79, 86, 35, 24, 31.$

[common elements cancelled]

Cyclic change in the order of the digits: Type 3.

Start with the equality $(10x+y)+(10z+w)=(10y+z)+(10w+x)$, where $x+z=y+w$ and $x,y,z,w < 9$. Apply Tarry's lemma with $d=11$ in succession (or a convenient multiple of 11 to avoid repetition of elements).

Example :—Let $x=1$, $y=3$, $z=4$, $w=2$.
We have, $13+42=34+21$

$\therefore$ $13, 42, 45, 32 \stackrel{2}{=} 34, 21, 24, 53.$ [taking $d=11$]

This can be rearranged in the form

$$13,\ 42,\ 45,\ 32 \overset{2}{=} 34,\ 24,\ 53,\ 21.$$

$$\therefore\ 13,\ 42,\ 45,\ 32,\ 56,\ 46,\ 75,\ 43 \overset{3}{=} 34,\ 24,\ 53,\ 21,\ 35,\ 64,\ 67,\ 54. \quad [d=22]$$

This can be rearranged in the form

$$13,\ 56,\ 42,\ 46,\ 75,\ 43,\ 45,\ 32 \overset{3}{=} 35,\ 64,\ 24,\ 67,\ 54,\ 34,\ 53,\ 21.$$

The types 1 and 2 can be generalised to numbers having k digits by starting with the numbers $a_1a_2 \ldots a_k$ and $b_1b_2 \ldots b_k$ where the digits a_i, b_i are such that $a_i + b_i =$ a constant, and writing their sum in the required form. Tarry's lemma can be applied to this equality in succession with $d = 1_{k-1}$ in the first case and 1_k in the second case, where $1_k = 111 \ldots 1$, the figure 1 being repeated k times. Here also repetition of an element can be avoided by choosing a convenient multiple of 1_{k-1} or 1_k.

Example :—Type 1.

$$3451 + 2104 = 3401 + 2154 \qquad [a_i + b_i = 5]$$

$$\therefore\ 3451,\ 2104,\ 3512,\ 2265 \overset{2}{=} 3562,\ 2215,\ 3401,\ 2154. \quad [d=111]$$

$$\therefore\ 3451,\ 2104,\ 3512,\ 2265,\ 3784,\ 2437,\ 3623,\ 2376$$
$$\overset{3}{=} 3673,\ 2326,\ 3734,\ 2487,\ 3562,\ 2215,\ 3401,\ 2154. \quad [d=222]$$

Example :—Type 2.

$$201 + 021 = 120 + 102. \qquad [a_i + b_i = 2]$$

$$\therefore\ 201,\ 021,\ 231,\ 213 \overset{2}{=} 312,\ 132,\ 120,\ 102. \quad [d=111]$$

$$\therefore\ 201, 021, 231, 213, 534, 354, 342, 324 \overset{3}{=} 423, 243, 453, 435, 312, 132, 120, 102. \quad [d=222]$$

It will be observed that $d = 1_{k-1}$ or 1_k is not applied in succession to avoid repetition of elements. Even if repetitions are allowed, the process followed fails to reach powers higher than 9. An attempt is made in the rest of this note to get high power equalities in the case of Type 2, where the elements on one side are obtained by reversing the digits of the elements on the other.

Let us apply Tarry's lemma with $d = 10$ in succession to the equality $001 + 110 = 011 + 100$.

$$\therefore\ 001,\ 110,\ 110,\ 021 \overset{2}{=} 120,\ 011,\ 011,\ 100.$$

i.e., $001,\ (110)_2,\ 021 \overset{2}{=} 120,\ (011)_2,\ 100.$

$$\therefore\ 001,\ (110)_3,\ (021)_3,\ 130 \overset{3}{=} 031,\ (120)_3,\ (011)_3,\ 100.$$

$$\therefore\ 001,\ (110)_4,\ (021)_6,\ (130)_4,\ 041 \overset{4}{=} 140,\ (031)_4,\ (120)_6,\ (011)_4,\ 100$$

$\therefore\quad 001,(110)_5,(021)_{10},(130)_{10},(041)_5,150 \overset{5}{=} 051,(140)_5,(031)_{10},(120)_{10},(011)_5,100.$

$\therefore\quad 001,(110)_6,(021)_{15},(130)_{20},(041)_{15},(150)_6,061$
$\overset{6}{=} 160,(051)_6,(140)_{15},(031)_{20},(120)_{15},(011)_6,100.$

$\therefore\quad 001,(110)_7,(021)_{21},(130)_{35},(041)_{35},(150)_{21},(061)_7,170$
$\overset{7}{=} 071,(160)_7,(051)_{21},(140)_{35},(031)_{35},(120)_{21},(011)_7,100.$

$\therefore\quad 001,(110)_8,(021)_{28},(130)_{56},(041)_{70},(150)_{56},(061)_{28},(170)_8,081$
$\overset{8}{=} 180,(071)_8,(160)_{28},(051)_{56},(140)_{70},(031)_{56},(120)_{28},(011)_8,100.$

$\therefore\quad 001,(110)_9,(021)_{36},(130)_{84},(041)_{126},(150)_{126},(061)_{84},(170)_{36},(081)_9,190$
$\overset{9}{=} 091,(180)_9,(071)_{36},(160)_{84},(051)_{126},(140)_{126},(031)_{84},(120)_{36},(011)_9,100.$

Further application of the lemma with $d=10$ is not possible, since the tens' digit of an element cannot be greater than 9. Before finding a way of circumventing this difficulty, let us note the simple order of succession in the elements of an equality and the order of succession in the repetitions of the elements.

Let us take the 7th degree equality to illustrate the law of formation. Let T denote the value of a number t when its digits are reversed. We have for the elements $t_1, t_2, \ldots, t_8$ of the 7th degree equality,

$$t_1=001 \qquad t_5=041=T_4+10$$
$$t_2=110=T_1+10 \qquad t_6=150=T_5+10$$
$$t_3=021=T_2+10 \qquad t_7=061=T_6+10$$
$$t_4=130=T_3+10 \qquad t_8=170=T_7+10.$$

Also, the number of repetitions of the elements of an equality obeys the law of binomial coefficients.

To reach powers greater than 9, let us apply Tarry's lemma to the 9th degree equality with $d=101$ in succession. The 10th degree equality is found to be

$$[001,(110)_9,(021)_{36},(130)_{84},(041)_{126},(150)_{126},(061)_{84},(170)_{36},(081)_9,190,$$
$$201,(112)_9,(221)_{36},(132)_{84},(241)_{126},(152)_{126},(261)_{84},(172)_{36},(281)_9,192.]$$
$$\overset{10}{=} [291,(182)_9,(271)_{36},(162)_{84},(251)_{126},(142)_{126},(231)_{84},(122)_{36},(211)_9,102,$$
$$091,(180)_9,(071)_{36},(160)_{84},(051)_{126},(140)_{126},(031)_{84},(120)_{36},(011)_9,100.$$

By applying the lemma with $d=101$ in succession it is therefore possible to reach as far as the 17th degree, when the process stops. This indicates the procedure for high power equalities.

Start with the first degree equation

$$0_{2n}1+1_{2n}0=01_{2n}+10_{2n}$$

where n is sufficiently large. The lemma has to be applied with the following values of d repeated as far as possible :—

$$0_n10_n,\ 0_{n-1}1010_{n-1},\ 0_{n-2}10_310_{n-2},\ \ldots,\ 0_{n-i}10_{2i-1}10_{n-i}\ \ldots,$$

the last application being with $d=10_{2n-1}1$. The order has to be observed so that one application may not produce complication with the maximum digit obtained by previous applications. All values of d except the last can be repeated 9 times while the last value of d can be repeated 8 times. In this way we can reach as far as $(9n+8)^{\text{th}}$ power.

The opening may be made also with the first degree equation

$$0_{2n-1}1+1_{2n-1}0=01_{2n-1}+10_{2n-1}.$$

In this case, the values of d in applying Tarry's lemma will be in the order $0_{n-1}1_20_{n-1}$, $0_{n-2}10_210_{n-2}$, $0_{n-3}10_410_{n-3}$, ... the last value of d being $10_{2n-2}1$. Thus it is possible to reach as far as $(9n+8)^{\text{th}}$ power when all the elements have an even number of digits.

Reversible Number Sets with Equal Sums of Powers*

BY

D. R. KAPREKAR, *Devlali*

1. In what follows, $a,b,c,\ldots$ denote digits. We write ab for $10a+b$, abc for $100a+10b+c$ and so on. The product of a and b is denoted by $a{\cdot}b$. A,B,L and R denote sets of positive integers.

We write

$$(\mathrm{A})^n=(\mathrm{B})^n$$

when the sums of the nth powers of the members of sets A and B are equal and

$$\mathrm{A} \overset{n}{=} \mathrm{B}$$

when $\qquad (\mathrm{A})^r=(\mathrm{B})^r,\ r=1,\ 2,\ 3,\ \ldots,\ n.$

The object of this paper is to construct numbers of the form :

$$abc\ldots k \text{ and } k\ldots cba$$

which have the digits in the reverse order and can be broken up into bits with equal sums of powers.

2. Theorem 1. If $(a_1,a_2,a_3,\ \ldots,\ a_j)^2=(b_1,b_2,b_3,\ \ldots,\ b_j)^2,\ j\geqslant 2$;

then $\qquad (a_1b_1,\ a_2b_2,\ a_3b_3,\ \ldots,\ a_jb_j)^2=(b_ja_j,\ \ldots,\ b_3a_3,\ b_2a_2,\ b_1a_1)^2.$

For $\qquad \sum(a_ib_i)^2=100\sum a_i^2+20\sum(a_i{\cdot}b_i)+\sum b_i^2=\sum(b_ia_i)^2;\ 1\leqslant i\leqslant j.$

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* My thanks are due to Dr. H. R. Gupta for presenting my work in this form.

In particular, if $a_1, a_2, a_3, \ldots, a_j \overset{2}{=} b_1, b_2, b_3, \ldots, b$; $j \geqslant 2$;

then $a_1b_1,\ a_2b_2,\ a_3b_3, \ldots, a_jb_j \overset{2}{=} b_ja_j,\ b_{j-1}a_{j-1}, \ldots, b_2a_2, b_1a_1.$

The converses of these results are also true.

Examples. We have $(3,4,8)^2 = (9,2,2)^2$,
therefore $(39,\ 42,\ 82)^2 = (28,\ 24,\ 93)^2$.

Again since $(9,\ 4,\ 6,\ 3) \overset{2}{=} (8,\ 7,\ 5,\ 2)$,
therefore $98,\ 47,\ 65,\ 32 \overset{2}{=} 23,\ 56,\ 74,\ 89.$

Many more examples can be constructed from results such as

$$(1,\ 8)^2 = (4,\ 7)^2 ;$$
$$7,\ 3,\ 2 \overset{2}{=} 6,\ 5,\ 1 ;$$
$$9,\ 6,\ 3,\ 2 \overset{2}{=} 8,\ 7,\ 4,\ 1 ;$$

and

$$8,\ 5,\ 3,\ 2 \overset{2}{=} 7,\ 6,\ 4,\ 1.$$

Theorem 2. $(a_1a_2,\ a_2a_3,\ a_3a_4, \ldots, a_ja_1) \overset{2}{=} (a_1a_j,\ a_ja_{j-1}, \ldots, a_3a_2,\ a_2a_1)$; $j \geqslant 3$.

This is a particular case of Theorem 1.

Example: $(12,\ 23,\ 31) \overset{2}{=} (13,\ 32,\ 21)$.

Theorem 3. $a,\ b0,\ ab \overset{2}{=} b,\ a0,\ ba.$

This is a particular case of Theorem 2, for we can write the result in the form: $0a,\ ab,\ b0 \overset{2}{=} 0b,\ ba,\ a0.$

Example: $3,\ 40,\ 34 \overset{2}{=} 4,\ 30,\ 43.$

Theorem 3 is of special importance to us, because applying Tarry's Lemma, we can obtain results for higher powers. Thus, taking $d=1$ in Tarry's Lemma, from the preceding example, we get

$$3,\ 40,\ 34,\ 5,\ 31,\ 44 \overset{3}{=} 4,\ 41,\ 35,\ 4,\ 30,\ 43$$

and

$$3,\ 40,\ 34,\ 5,\ 31,\ 44,\ 5,\ 31,\ 44,\ 5,\ 42,\ 36$$
$$\overset{4}{=} 6,\ 32,\ 45,\ 4,\ 41,\ 35,\ 4,\ 41,\ 35,\ 4,\ 30,\ 43.$$

Theorem 4. If $a_2 \cdot a_3 + a_4 \cdot a_5 + a_6 \cdot a_7 + \cdots + a_{j-1} \cdot a_j$
$= a_1 \cdot a_2 + a_3 \cdot a_4 + a_5 \cdot a_6 + \cdots + a_{j-2} \cdot a_{j-1},\ j > 4$;

then $a_1,\ a_2a_3,\ a_4a_5, \ldots, a_{j-1}a_j \overset{2}{=} a,\ a_{j-1}\,a_{j-2}, \ldots, a_6a_5,\ a_4a_3,\ a_2a_1$;
j odd and conversely.

The theorem is easy to prove.

Examples: $2,\ 44,\ 26,\ 43 \overset{2}{=} 3,\ 46,\ 24,\ 42$
because $4 \cdot 4 + 2 \cdot 6 + 4 \cdot 3 = 4 \cdot 6 + 2 \cdot 4 + 4 \cdot 2.$

Again $3,\ 34,\ 51,\ 27 \overset{2}{=} 7,\ 21,\ 54,\ 33$
because $3 \cdot 4 + 5 \cdot 1 + 2 \cdot 7 = 2 \cdot 1 + 5 \cdot 4 + 3 \cdot 3.$

Tarry's Lemma is applicable to these results. The results cannot, however, be extended beyond fourth powers in the latter and the fifth powers in the former case. In fact, in order that it may be possible to extend a result to higher powers, it is necessary that the digits in the units places be small. Thus, from

$$2,\ 10,\ 21 \overset{2}{=} 1,\ 20,\ 12 \tag{1}$$

we would be able to get results up to 9th powers by a repeated use of Tarry's Lemma with $d=1$ at each stage.

3. To find the form of the answer for any power $\leqslant 9$, we denote the set of elements on the left of (1) by L and that on the right by R. We denote by A_{+k} the set obtained by adding k to each of the members of A and by $(A)_m$ the set of elements obtained by repeating the set A, m times. From (1), we then have

$$L,\ R_{+1} \overset{3}{=} L_{+1},\ R\ ;$$

$$L,\ (R_{+1})_2,\ (L_{+2}) \overset{4}{=} (R_{+2}),\ (L_{+1})_2,\ R\ ;$$

$$L,\ (R_{+1})_3,\ (L_{+2})_3,\ (R_{+3}) \overset{5}{=} (L_{+3}),\ (R_{+2})_3,\ (L_{+1})_3,\ R\ ; \text{ and so on.}$$

The law governing the repetition of sets is that of Binomial coefficients. For 9th powers, we would have 192 elements and 320 digits on each side.

4. We add here the following interesting example which provides with a peculiar decomposition of the number $(5)_{24}$ i.e., the digit 5 repeated 24 times.

Starting from the top of a circle, and going clockwise round it, let us mark the numbers

04 20 34 12 50 42 03 41 53 15 31 25. (I)

Reading these numbers in the anti-clockwise direction, we get

05 21 35 13 51 43 02 40 52 14 30 24. (II)

Then, we have

$$(I) \overset{3}{=} (II).$$

Arranging the numbers (I) in "ascending" order (first arranging the left side numbers, and then the right side ones), we get

03 04 12 15 20 25 31 34 41 42 50 53. (III)

Similarly, arranging (II) in "descending" order, we get

52 51 43 40 35 30 24 21 14 13 05 02. (IV)

Adding (III) and (IV), we obtain $(5)_{24}$.

On the number defined by $N_r = \frac{1}{e} \sum_{n=0}^{\infty} \frac{n^r}{n!}$

BY

N. BALASUBRAHMANIAN

1. It is well-known that $\sum_{n=0}^{\infty} \frac{n^r}{n!} = N_r e$, where $N_r = \sum_{s=0}^{r-1} A_s$, the A's being defined by the identity
$n^{r-1} \equiv A_0 + A_1(n-1) + A_2(n-1)(n-2) + \cdots + A_{r-1}(n-1)(n-2)\cdots(n-r+1)$.
r is assumed to be a positive integer, and N_r is necessarily a positive integer.

Let $F(x) \equiv e^{e^x} = 1 + e^x + \frac{e^{2x}}{2!} + \frac{e^{3x}}{3!} + \cdots$.

The coefficient of $\frac{x^r}{r!}$ in the expansion of $F(x)$ is therefore $\sum_{n=0}^{\infty} \frac{n^r}{n!} = N_r e$. Using Maclaurin's theorem, it follows that

$$N_r e = F^{(r)}(0). \tag{1}$$

Writing down $F'(x)$, $F''(x)$, etc., we readily obtain from (1),

$$N_0 = 1,\ N_1 = 1,\ N_2 = 2,\ N_3 = 5, \text{ etc.} \tag{2}$$

$F^{(r)}(x)$ is equal to $F(x)$ multiplied by a polynomial of degree r in e^x. We write

$$F^{(r)}(x) = F(x) \sum_{s=1}^{r} S_{r,s}\, e^{sx}. \tag{3}$$

Differentiating,

$$F^{r+1}(x) = F(x)[S_{r,1}\, e^x + (S_{r,1} + 2\, S_{r,2})e^{2x} + \cdots + (S_{r,s-1} + sS_{r,s})e^{sx} + \cdots + S_{r,r}e^{(r+1)x}]. \tag{4}$$

Also, from (3), $$F^{r+1}(x) = F(x) \sum_{s=1}^{r+1} S_{r+1,s}\, e^{sx}. \tag{5}$$

From (4) and (5), we obtain

$$\left.\begin{aligned} &S_{r+1,s} = S_{r,s-1} + sS_{r,s}. \\ &S_{r+1,1} = S_{r,1},\ S_{r,r} = S_{r+1,\,r+1} \\ &S_{r,s} = 0 \text{ for } s > r;\ S_{r,0} = 0. \end{aligned}\right\} \tag{6}$$

Putting $x = 0$ in (3) and (4), we obtain

$$\left.\begin{aligned} N_r &= S_{r,1} + S_{r,2} + \cdots + S_{r,r} \\ &= 2S_{r-1,1} + 3S_{r-1,2} + 4S_{r-1,3} + \cdots + rS_{r-1,\,r-1}. \end{aligned}\right\} \tag{7}$$

From the values of $F'(x)$, $F''(x)$, etc. we readily find

$$S_{1,1} = 1 = S_{2,1} = S_{2,2}$$
$$S_{3,1} = 1,\ S_{3,2} = 3,\ S_{3,3} = 1.$$

The values of $S_{4,s}$, $S_{5,s}$ etc. can be computed by using (6).

2. Writing

$$N_r = \frac{1}{e}\sum_{r=1}^{\infty}\frac{(1+\overline{n-1})^{r-1}}{(n-1)!} = \frac{1}{e}\sum_{r=1}^{\infty}\sum_{s=0}^{r-1}\frac{{}_{r-1}C_s(n-1)^s}{(n-1)!},$$

we have

$$N_r = \sum_{s=0}^{r-1} {}_{r-1}C_s N_s. \tag{8}$$

This can also be deduced by differentiating both sides of $F'(x)=e^xF(x)$, $r-1$ times and putting $x=0$. From (8), we have

$$N_r - N_{r-1} = N_0 + {}_{r-1}C_1N_1 + {}_{r-1}C_2N_2 + \cdots + {}_{r-1}C_{r-2}N_{r-2}$$
$$N_{r-1} = N_0 + {}_{r-2}C_1N_1 + \cdots + {}_{r-2}C_{r-2}N_{r-2}.$$

Hence, $N_r - 2N_{r-1} = f(r-2)$,

where $f(r) = {}_rC_0N_1 + {}_rC_1N_2 + \cdots + {}_rC_{r-1}N_r$.

Hence, $N_r = f(r-2) + 2f(r-3) + 2^2f(r-4) + \cdots + 2^{r-3}f(1) + 2^{r-1}$. (9)

From (6), we easily get

$$S_{r+1,s} = sS_{r,s} + (s-1)S_{r-1,\,s-1} + (s-2)S_{r-2,\,s-2} + \cdots + S_{r-s+1,1} \tag{10}$$

and also $= S_{r,s-1} + sS_{r-1,\,s-1} + s^2S_{r-2,\,s-1} + \cdots + s^{r+1-s}S_{s-1,\,s}$. (11)

From (7) and (6), we obtain, taking $S_{r,s}=0$ when $s>r$, and for $r<1$, $s<1$,

$$\begin{aligned}
N_r &= \sum_{s=1}^{r-1}(s+1)S_{r-1,s} = \sum_{s=1}^{r-1}(s+1)\ [S_{r-2,\,s-1}+sS_{r-2,s}] \\
&= \sum_{s=1}^{r-2}(s+2)S_{r-2,s} + \sum_{s=1}^{r-1}s(s+1)S_{r-2,s} \\
&= \sum_{s=1}^{r-2}(s^2+2s+2)S_{r-2,s} \\
&= \sum_{s=1}^{r-2}(s^2+2s+2)\ [S_{r-3,\,s-1}+sS_{r-3,s}] \\
&= \sum_{s=1}^{r-3}[(s+1)^2+2(s+1)+2]S_{r-3,\,s-1} + \sum_{s=1}^{r-3}s(s^2+2s+2)S_{r-3,\,-1} \\
&\equiv \sum_{s=1}^{r-3}[(s+1)^2+s^3]S_{r-3,\,s-1} \pmod 2 \\
&\equiv \sum_{s=1}^{r-3}[s(s+1)+s(s^2+1)+1]S_{r-3,\,s-1} \pmod 2 \\
&\equiv \sum_{s=1}^{r-3}[s(s^2+1)+1]S_{r-3,\,s-1} \pmod 2 \\
&\equiv \sum_{s=1}^{r-3}S_{r-3,s} \pmod 2,\ \text{(change } s-1 \text{ to } s) \\
&= N_{r-3} \pmod 2.
\end{aligned}$$

But $N_1=1$, $N_2=2$, $N_3=5$. It follows that

Theorem. N_r *is even when* $r\equiv 2 \pmod 3$, *and odd when* $r\equiv 0$ or $1 \pmod 3$.

Next, from (8), we get

$$\begin{aligned} N_r + 2^{r-1} &= ({}_{r-1}C_0N_0 + {}_{r-1}C_1N_1 + \cdots + {}_{r-1}C_{r-1}N_{r-1}) \\ &\qquad + ({}_{r-1}C_0 + {}_{r-1}C_1 + \cdots + {}_{r-1}C_{r-1}) \\ &= [{}_{r-1}C_0(N_0+1) + {}_{r-1}C_1(N_1+1) + {}_{r-1}C_2N_2 + {}_{r-1}C_3(N_3+1) \\ &\quad + {}_{r-1}C_4(N_4+1) + {}_{r-1}C_5N_5 + \cdots] + [{}_{r-1}C_2 + {}_{r-1}C_5 + {}_{r-1}C_8 + \cdots] \\ &\equiv ({}_{r-1}C_2 + {}_{r-1}C_5 + {}_{r-1}C_8 + \cdots) \bmod 2, \text{ by the above theorem.} \end{aligned}$$

Hence,

Theorem :

$${}_{3n}C_2 + {}_{3n}C_5 + {}_{3n}C_8 + \cdots + {}_{3n}C_{3n-1} \textit{ is odd}$$
$${}_{3n-1}C_2 + {}_{3n-1}C_5 + \cdots + {}_{3n-1}C_{3n-1} \textit{ is odd}$$
$${}_{3n+1}C_2 + {}_{3n+1}C_5 + \cdots + {}_{3n+1}C_{3n+1} \textit{ is even.}$$

Some properties of the series $\sum_{n=0}^{\infty} \frac{(n+a)^r x^n}{n!}$

BY

S. N. MANIKARNIKAMMA, *Bangalore.*

We know that $\frac{1}{e^x}\sum_0^\infty \frac{P_r(n)x^n}{n!}$ is a polynomial of degree r in x when $P_r(n)$ is a polynomial of degree r in n, for, we can express $P_r(n)$ in the form

$$P_r(n) \equiv p_0 + p_1 n + p_2 n(n-1) + \cdots + p_r n(n-1)\cdots(n-r+1) \text{ and}$$
$$\sum_0^\infty \frac{P_r(n)x^n}{n!} = p_0\sum_0^\infty \frac{x^n}{n!} + p_1\sum_1^\infty \frac{x^n}{(n-1)!} + p_2\sum_2^\infty \frac{x^n}{(n-2)!} + \cdots + p_r\sum_r^\infty \frac{x^n}{(n-r)!}$$
$$= [p_0 + p_1x + p_2x^2 + \cdots + p_rx^r]e^x.$$

1. Consider the particular case $P_r(n) = (n+a)^r$.

Put $(n+a)^r = \sum_{s=0}^{r} A_{r,s}n\cdot(n-1)\cdots(n-s+1) = n!\sum_{s=0}^{r} A_{r,s}\cdot\frac{1}{(n+s)!}$. (1)

$$\sum_0^\infty \frac{(n+a)^r}{n!}x^n = f_r(x)e^x = e^x[A_{r,0} + A_{r,1}x + A_{r,2}x^2 + \cdots + A_{r,r}x^r]. \quad (2)$$

Since the L.H.S. series is uniformly convergent after any number of term by term differentiations, differentiating (2), we easily get

$$f_{r+1}(x) = (x+a)f_r(x) + xf'_r(x). \quad (3)$$

i.e.,
$$\left.\begin{aligned} A_{r+1,s} &= (a+s)A_{r,s} + A_{r,s-1} \\ A_{r,0} = a^r,\ A_{r,r} &= 1,\ A_{r,s} = 0 \text{ for } s > r \end{aligned}\right\}. \tag{4}$$

We easily see that $A_{r,s}$ is a polynomial of degree $(r-s)$ in a, and when a is an integer, $A_{r,s}$ is also an integer. Equations (4) enable us to calculate $A_{r,s}$ for all values of r and s.

Now $\sum_0^\infty \frac{(n+a)^r x^n}{n!} = \sum_0^\infty \frac{(n+a)(n+a)^{r-1}x^n}{n!}$

$$= x\sum_1^\infty \frac{(\overline{n-1+a}+1)^{r-1}x^{n-1}}{(n-1)!} + a\sum_0^\infty \frac{(n+a)^{r-1}}{n!}x^n$$

$$= x\left[\sum_{s=0}^{r-1} {}_{r-1}C_s f_s\right] e^x + af_{r-1}e^x.$$

$$\therefore\quad f_r(x) = x[{}_{r-1}C_0 f_0 + {}_{r-1}C_1 f_1 + \cdots + {}_{r-1}C_{r-1} f_{r-1}] + af_{r-1}. \tag{5}$$

From (4), $A_{r,s} = A_{r-1,s-1} + (s+a)A_{r-1,s}$
$= A_{r-1,s-1} + (s+a)[A_{r-2,s-1} + (s+a)A_{r-2,s}]$ etc.

So that we have

$$A_{r,s} = A_{r-1,s-1} + (s+a)A_{r-2,s-1} + (s+a)^2A_{r-3,s-1} + \cdots + A_{s,s-1}(s+a)^{r-s-1} + (s+a)^{r-s}. \tag{6}$$

Also $A_{r,s} = (s+a)A_{r-1,s} + A_{r-1,s-1}$
$= (s+a)A_{r-1,s} + (s+a-1)A_{r-2,s-1} + A_{r-2,s-2}$ and so on.

$$\therefore\quad A_{r,s} = (s+a)A_{r-1,s} + (s+a-1)A_{r-2,s-1} + (s+a-2)A_{r-3,s-2} + \cdots$$
$$\cdots + (a+2)A_{r-\ +1,2} + (a+1)A_{r-s,1} + A_{r-s,0}. \tag{7}$$

2. $$f_r(x) = \sum_{s=0}^{r} A_{r,s}x^s = \sum_{s=0}^{r} [(a+s)A_{r-1,s} + A_{r-1,s-1}]x^s$$

$$= \sum_{s=0}^{r-1} (a+s)A_{r-1,s}x^s + \sum_{s=0}^{r-1} A_{r-1,s}x^{s+1}$$

$$= \sum_{s=0}^{r-1} (a+s+x)A_{r-1,s}x^s.$$

After two more such reductions, we get

$$f_r(x) = \sum_{s=0}^{r-3} \{(s+a)(a^2+2as+2ax+s^2+2xs+x+x^2) + (a+s+x+1)(a+s+1)x + x^3 + 2x^2 + sx^2 + ax^2\}A_{r-3,s}x^s.$$

(i) When x is an odd integer and a any integer,

$$f_r(x) \equiv \sum_{s=0}^{r-3} A_{r-3,s}x^s \pmod 2 \text{ for all values of } r.$$

$$\therefore\quad f_r(x) \equiv f_{r-3}(x) \pmod 2. \tag{8}$$

and from (3), we have $f_1(x)=(x+a)$, $f_2(x)=(x+a)^2+x$

$f_3(x)=(x+a)^3+3x(x+a)+x$, so that we have:

When a and x are integers, x being odd,

$$\left.\begin{array}{ll} f_1\equiv(a+1)(\text{mod } 2) & \text{i.e., } f_{3r+1}\equiv a+1(\text{mod } 2) \\ f_2\equiv \quad a \quad (\text{mod } 2) & \text{i.e., } f_{3r+2}\equiv a \quad (\text{mod } 2) \\ f_3\equiv \quad 1 \quad (\text{mod } 2) & \text{i.e., } f_{3r} \quad \equiv 1 \quad (\text{mod } 2) \end{array}\right\} \qquad (9a)$$

(ii) If x is even and a any integer,

$f_r(x)\equiv A_{r,0}(\text{mod } 2)\equiv a^r\equiv a(\text{mod } 2)$

$\therefore \quad f_r(x)\equiv a \; (\text{mod } 2)$ for all r. (9b)

3. Let us put $N_r(x)=\frac{1}{e^x}\sum_0^\infty \frac{n^r x^n}{n!}$.

i.e., $N_r(x)$ is got from $f_r(x)$ by putting $a=0$.

$$N_r(x)=\sum_{s=1}^{r} B_{r,s}x^s$$

where $n^{r-1}=B_{r,1}+B_{r,2}(n-1)+B_{r,3}(n-1)(n-2)+\cdots+B_{r,r}(n-1)\cdots(n-r+1)$. (10)

In addition to (4), $B_{r,s}$ satisfies the relations

$$B_{r,0}=0,\; B_{r,1}=1 \text{ for all } r. \qquad (10a)$$

$$e^x f_r(x)=\sum_0^\infty \frac{(a+n)^r x^n}{n!}=\sum_0^\infty\sum_{s=0}^{r}\frac{{}_rC_s a^{r-s}n^s x^n}{n!}=e^x\sum_{s=0}^{r}{}_rC_s N_s(x)a^{r-s}$$

$$\therefore \quad f_r(x)=\sum_{s=0}^{r}{}_rC_s a^{r-s}N_s(x)=\sum_{s=0}^{r}\sum_{t=1}^{s}{}_rC_s a^{r-s}B_{s,t}\,x^t. \qquad (11)$$

$\therefore \quad f_r(x)$ is a polynomial of degree r in a.

Coefficient of x^s in $f_r(x)$ is $A_{r,s}=\sum_{l=s}^{r}{}_rC_l\cdot B_{l,s}\,a^{r-l}$

$$\therefore \quad A_{r,s}={}_rC_0\cdot B_{r,s}+{}_rC_1\cdot B_{r-1,s}\cdot a+{}_rC_2\cdot B_{r-2,s}\cdot a^2+\cdots+{}_rC_{r-s}B_{s,s}\,a^{r-s}. \qquad (12)$$

If $P_{r,s}$ is the coefficient of a^s in $f_r(x)$, $P_{r,s}={}_rC_sN_{r-s}(x)$

$$\therefore \quad P_{r,s}=r/s\cdot P_{r-1,\,s-1}. \qquad (13)$$

Since $B_{r,1}=1$, from (10), $B_{r,2}-B_{r,3}\cdot 2!+B_{r,4}\cdot 4!-+\cdots$

$$+(-1)^n B_{r,r}\cdot n!=1. \qquad (14)$$

From (12) $A_{r,1}=(1+a)^r-a^r$ for all r. (15)

4. From (4) we can write down $A_{r,s}$ for all r and s, and in particular, when $a=0$, we get $B_{r,s}$, from which we can calculate $N_r(1)$. The expression $N_r(1)=N_r$ is identical with the function considered in the previous note by N. Balasubrahmanian.

From (9) we readily deduce Balasubrahmanian's result that

N_r is odd if $r \equiv 0$ or 1 (mod 2)

and even if $r \equiv 2$ (mod 2).

If $F(x) = e^{e^x} \equiv 1 + e^x + \dfrac{e^{2x}}{2!} + \cdots,$

it is verified that $F^{(r)}(x) = F(x) \sum_{s=1}^{r} B_{r,s} e^{sx}$, showing that $B_{r,s}$ is identical with $S_{r,s}$ of N. Balasubrahmanian's paper.

Next, $N_r(e^x) = \sum_{s=1}^{r} B_{r,s} e^{sx} = F^{(r)}(1) \cdot \dfrac{1}{e^{e^x}}$ so that we deduce,

$$\left(\frac{d}{dx}\right)^r \left[e^{e^x}\right] = e^{e^x} N_r(e^x) = \sum_{1}^{\infty} \frac{n^r e^{nx}}{n!}. \tag{16}$$

Further, $\displaystyle\sum_{n=0}^{\infty} \frac{(n+a)^r e^{nx}}{n!} = \sum_{n=0}^{\infty} \sum_{s=0}^{r} {}_rC_s \cdot n^s \cdot a^{r-s} \cdot e^{nx} \cdot \frac{1}{n!}$

$$= \sum_{s=0}^{r} {}_rC_s \cdot a^{r-s} \left(\frac{d}{dx}\right)^s \left(e^{e^x}\right), \text{ by (16)},$$

$$= \frac{1}{e^{ax}} \cdot \sum_{s=0}^{r} {}_rC_s e^{ax} \cdot a^{r-s} \cdot \left(\frac{d}{dx}\right)^s \left(e^{e^x}\right)$$

$$\sum_{n=0}^{\infty} \frac{(n+a)^r e^{nx}}{n!} = \frac{1}{e^{ax}} \left(\frac{d}{dx}\right)^r \left(e^{e^x} \cdot e^{ax}\right)$$

$$\therefore \quad \left(\frac{d}{dx}\right)^r \left[e^{(ax+e^x)}\right] = e^{ax} \sum_{n=0}^{\infty} \frac{(n+a)^r \cdot e^{nx}}{n!} \cdots \tag{17}$$

Thus, incidentally, we arrive at the two neat identities (16) and (17).

A note on 'Four Fours'

BY

G. C. PATNI, *Jaipur*.

The problem of expressing all positive integers by the repeated use of a single digit like 4 is an interesting mathematical recreation. Many well-known puzzle makers have worked on it by using the arithmetical symbols +, −, ×, ÷, √, ·, !, fourth power, $\sqrt[4]{}$ (fourth root) etc. Recently Mazumdar[1] has been able to express all natural numbers up to 1,000 and some integers beyond that, in terms of 'four fours', by using a new symbol $\sum$ in the sense $\sum 4 = 1 + 2 + 3 + 4 = 10$.

By the use of new symbols, I have been able to express most of the numbers up to 10,000 and many more beyond that, by means of four 4's.

The new symbols used by me are:—

(i) $[n]$ = the integral part of n.

(ii) $p(n)$ = number of unrestricted partitions of n.

Thus[2] $p(4)=5$; $p(8)=p(4+4)=22$; $p(22)=p(4\,!-\sqrt{4})=1002$ etc.

(iii) $\overset{k}{\sum}(n)=1^k+2^k+3^k+\cdots\cdots+n^k$ = sum of kth powers of the first n natural numbers.

Thus when $k=1$, $\overset{k}{\sum}(n)=\sum(n)=\dfrac{n(n+1)}{2}$;

$$k=2,\ \overset{k}{\sum}(n)=\overset{\sqrt{4}}{\sum}(n)=\frac{n(n+1)(2n+1)}{6}$$

$$k=3,\ \overset{k}{\sum}(n)=\overset{(\Sigma\sqrt{4})}{\sum}(n)=\left\{\frac{n(n+1)}{2}\right\}^2$$

$$k=4,\ \overset{k}{\sum}(n)=\overset{4}{\sum}(n)=\frac{n(n+1)(6n^3+9n^2+n-1)}{30}$$

and so on. Thus $\overset{\sqrt{4}}{\sum}(4)=30$, $\overset{4}{\sum}(\sum\sqrt{4})=98$ etc.

(iv) $\sigma_s(n)$ = sum of the sth powers of the divisors of n (including 1 and n).

Thus when $s=1$, $\sigma_s(n)=\sigma(n)$ = sum of the divisors of n is given by the formula[3]

$$\sigma(n)=\frac{(p_1{}^{\alpha_1+1}-1)(p_2{}^{\alpha_2+1}-1)\cdots(p_k{}^{\alpha_k+1}-1)}{(p_1-1)(p_2-1)\cdots(p_k-1)}.$$

where $n=p_1{}^{\alpha_1}p_2{}^{\alpha_2}\cdots p_k{}^{\alpha_k}$; $p_1, p_2\cdots p_k$ being all primes.

Thus $\sigma(\sum 4)=\sigma(10)=18$, $\sigma(4+4)=\sigma(8)=15$ etc.

(v) $\phi(n)$ = number of integers (including 1) which are less than n and prime to n and is given by

$$\phi(n)=n\left(1-\frac{1}{a_1}\right)\left(1-\frac{1}{a_2}\right)\left(1-\frac{1}{a_3}\right)\cdots\left(1-\frac{1}{a_k}\right)\text{ where } n=a_1{}^{\alpha_1}a_2{}^{\alpha_2}\cdots a_k{}^{\alpha_k}.$$

By using these symbols and working on similar lines, it is possible to make further progress in the matter. Some examples from my list are given here by way of illustrations :—

$$1024=p(4\,!-\sqrt{4})+p(4\times\sqrt{4})$$
$$1062=p(4\,!-\sqrt{4})+(\sum 4)\times\{\sum(\sum\sqrt{4})\}$$
$$1137=p(4\,!-\sqrt{4})+p(\sum 4+4)$$
$$1334=p[\sum\{\sum(\sum\sqrt{4})\}+\sqrt{4}]+\sum(\sum 4)+4\,!$$
$$1399=p[\sum\{\sum(\sum\sqrt{4})\}+\sqrt{4}]+4\,!\times(\sum\sqrt{4})\,!$$
$$1639=p(4\,!)+4+4+4$$
$$1717=p[\sum\{\sum(\sum\sqrt{4})\}+\sqrt{4}]+\sum[\sum\{\sum(\sum\sqrt{4})\}]\times 4\,!$$

$$1831 = p[\sum\{\sum(\sum\sqrt{4})\} + \sqrt{4}] + (4\,!) \times (4\,!)$$

$$2489 = p(4\,! + \sqrt{4}) + \sum(\sum 4) - \sqrt{4}$$

$$3756 = p(4\,! + 4) + \sqrt{4} \times 4\,!$$

$$4560 = \phi[\{\sum(4\,!)\} \times 4\,!] \times \phi\{\sum(\sum\sqrt{4})\} + \overline{\{\sum(\sum\sqrt{4})\}}\,!$$

$$6962 = p[\sum\{\sum(\sum\sqrt{4})\} + \sum 4] + \phi[\sum\{\sum\overline{\sum(\sum\sqrt{4})}\}]$$

$$7227 = p[\sum\{\sum(\sum\sqrt{4})\} + \sum 4] + \frac{\sum\{\sum(\sum 4)\}}{4}$$

$$9604 = \{\sum^{4}(\sum\sqrt{4})\} \times \{\sum^{4}(\sum\sqrt{4})\}$$

$$12{,}408 = p(4\,! + \sum 4) + \sum^{4}(\sum\sqrt{4})$$

$$37{,}723 = p(\sum 4 \times 4) + \sum^{\sqrt{4}}(\sum 4)$$

$$241{,}992 = \phi[\{\sum(\sum 4\,!)\} \times 4\,!] + (4\,! \times \sum\sqrt{4})$$

$$294{,}601 = p(4\,! \times \sqrt{4}) \times \phi(\sum\sqrt{4}) + \sum\sum 4$$

$$1{,}919{,}541 = p\{\sum(\sum 4) - 4\} \times \phi(4\,!) - \sum\sqrt{4}$$

$$5{,}540{,}056 = p\{4\,! \times \sum\sqrt{4}\} + p(\sqrt{4} \times 4\,!)$$

$$22{,}658{,}768{,}749 = p\{4\,! \times \sum(\sum\sqrt{4})\} + p(4 \times 4\,!) \text{ etc.}$$

By means of the new symbols adopted here, we can now easily express even 29 in terms of 'two fours' which Mazumdar could not do with his symbols, e.g.,

$$29 = 4\,! + p(4), \text{ or } 29 = \sigma(\sum 4) + p\{\sum(\sum\sqrt{4})\}$$
$$\text{or} = \phi(4\,!) + \sum\{\sum(\sum\sqrt{4})\}.$$

References

1. S. V. Mazumdar, *Math. Student*, 1947, 15, pp. 48-49.

2. *Collected Papers of S. Ramanujan* by Hardy, Aiyar and Wilson—Table IV of MacMahon p. 308. *Tables of Partitions* by Dr. Hansraj Gupta.

3. *Algebra* by G. Chrystal. Pt. II. Second Edition, p. 537.

SOME INTERESTING PROBLEMS AND THEIR SOLUTIONS

(A) BY T. S. NANJUNDIAH, *Bangalore*

Problem 1. *For $n \to \infty$, establish the following asymptotic formula:*

$$\sum_{r=0}^{n} \binom{n}{r} \frac{\zeta^r}{2r+1} \sim \begin{cases} \dfrac{(1+\zeta)^{n+1}}{2n\zeta} & \text{for } |1+\zeta|>1, \\ \dfrac{1}{2}\sqrt{\dfrac{\pi}{-n\zeta}} & \text{for } |1+\zeta| \leqq 1 \ (\zeta \neq 0), \end{cases} \qquad (\zeta \text{ complex}).$$

Solution: We first establish the identity:

$$\text{(I)} \quad \sum_{r=0}^{n} \binom{n}{r} \frac{\zeta^r}{2r+1} = \frac{1}{(2n+1)g_n} \sum_{r=0}^{n} g_r (1+\zeta)^r$$

with $g_0 = 1$, $\qquad g_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \qquad (n \geqq 1).$

Note that the given sum can be represented as the integral

$$I_n = \frac{1}{\sqrt{\zeta}} \int_0^{\sqrt{\zeta}} (1+z^2)^n \, dz \qquad (n \geqq 0).$$

Integration by parts yields the following recursion for I_n:

$$(2n+1)\, I_n = 2n\, I_{n-1} + (1+\zeta)^n \qquad (n \geqq 1),$$

which easily leads to (I).

(*a*) Let $|1+\zeta|>1$: writing (I) in the form

$$I_n = \frac{(1+\zeta)^n}{2n+1}\, U_n,$$

with $\qquad U_n = \sum_{r=0}^{n} u_{r,n}, \qquad u_{r,n} = \dfrac{g_{n-r}}{g_n} \dfrac{1}{(1+\zeta)^r},$

we have only to show that

$$\lim_{n \to \infty} U_n = \frac{1+\zeta}{\zeta}.$$

We prove this by using Tannery's Theorem. We have

$$(1) \quad \lim_{n \to \infty} u_{r,n} = \frac{1}{(1+\zeta)^r} \qquad \text{for every fixed } r.$$

To get a suitable bound above for $|u_{r,n}|$ required by Tannery's Theorem, we first note that $\dfrac{g_{n-r}}{g_n}$ is a decreasing function of n for $n \geqq r$, this being true for $r=1$. It follows that $\dfrac{g_{n-r}}{g_n} < \dfrac{1}{g_r}$ $(n>r>0)$, giving

(2) $\quad | u_{r,n} | \leqq M_r \equiv \dfrac{1}{g_r \, |1+\zeta|^r} \qquad (0 \leqq r \leqq n).$

Also, D'Alembert's test shows at once that

(3) $\quad \sum\limits_0^\infty M_r$ is convergent.

Thus the conditions of Tannery's Theorem are satisfied. Hence
$\lim\limits_{n\to\infty} U_n = \sum\limits_{r=0}^{\infty} \dfrac{1}{(1+\zeta)^r} = \dfrac{1+\zeta}{\zeta}$, as required.

(*b*) Let $|1+\zeta| \leqq 1 \; (\zeta \neq 0)$: in this case, (I) yields at once

$$\lim_{n\to\infty} \quad (2n+1) \, g_n \, I_n = \{1-(1+\zeta)\}^{-\frac{1}{2}},$$

giving the required result on using $g_n \sim \dfrac{1}{\sqrt{n\pi}}$.

Cor. $\displaystyle \lim_{n\to\infty} \frac{I_{n+1}}{I_n} = \begin{cases} 1+\zeta & \text{for } |1+\zeta| > 1, \\ 1 & \text{for } |1+\zeta| \leqq 1, \end{cases} \qquad (\zeta \text{ complex}).$

The problem was suggested by the case $\zeta = 1$ of this corollary, which case appeared in the *Amer. Math. Monthly*, 57, 1, Prob. 4378.

Problem 2. *Given* $d_n > 0$, $\sum d_n$ *divergent, discuss the behaviour of the series* $\sum \dfrac{d_n}{1+n^\rho d_n}$, $(-\infty < \rho < +\infty)$.

[See Knopp: *Theory and Application of Infinite Series* p. 125, ex. 35 ($\rho = 0, 1, 2$)].

Solution: We shall establish the following result:

The series in question converges for $\rho > 1$, diverges for $\rho \leqq 0$ and behaves indefinitely for $0 < \rho \leqq 1$. Moreover, the last behaviour on the part of our series is exhibited only when $n^\rho d_n$ has 0 and $+\infty$ as its extreme limits, the series being otherwise divergent.

I. Let $\rho > 1$: Since $\dfrac{d_n}{1+n^\rho d_n} < \dfrac{1}{n^\rho}$, our series converges by comparison with $\sum \dfrac{1}{n^\rho}$ $(\rho > 1)$.

II. Let $\rho \leqq 0$: In this case, $\dfrac{d_n}{1+n^\rho d_n} \geqq \dfrac{d_n}{1+d_n}$ so that it is enough if we show that $\sum \dfrac{d_n}{1+d_n}$ is divergent.

There are only two possibilities:

(a) $\overline{\lim}\, d_n < +\infty$, (b) $\overline{\lim}\, d_n = +\infty$.

In case (a), d_n is bounded above: say, $d_n < \Delta$ for all n. Then, $\frac{d_n}{1+d_n} > \frac{d_n}{1+\Delta}$ for all n, whence $\sum \frac{d_n}{1+d_n}$ diverges by comparison with $\sum d_n$.

In case (b), $d_n > 1$ for infinitely many n. For these n, $\frac{d_n}{1+d_n} > \frac{1}{2}$. Hence $\sum \frac{d_n}{1+d_n}$, being a positive series, diverges.

III. Let $0 < \rho \leqq 1$: There are only three possibilities, namely

(a) $\underline{\lim}\, n^\rho d_n > 0$, (b) $\overline{\lim}\, n^\rho d_n < +\infty$

(c) $\underline{\lim}\, n^\rho d_n = 0$ and $\overline{\lim}\, n^\rho d_n = +\infty$.

In case (a), choose a δ with $0 < \delta < \underline{\lim}\, n^\rho d_n$. Then, for a certain n_0, $n^\rho d_n > \delta$ $(n \geqq n_0)$. Consequently $\frac{n^\rho d_n}{1+n^\rho d_n} > \frac{\delta}{1+\delta}$ $(n \geqq n_0)$, so that our series diverges by comparison with $\sum \frac{1}{n^\rho}$ $(\rho \leqq 1)$.

In case (b), $n^\rho d_n$ is bounded above: say, $n^\rho d_n < \Delta$ for all n. Then, $\frac{d_n}{1+n^\rho d_n} > \frac{d_n}{1+\Delta}$ for all n, whence our series diverges by comparison with $\sum d_n$. In the critical case (c), our series may diverge or converge as the following two examples illustrate:

(1) Let $d_n = \begin{cases} \frac{1}{n^{1+\rho}}, & n \text{ odd}, \\ n^{1-\rho}, & n \text{ even}. \end{cases}$

Plainly, this comes under case (c).

We have $\frac{d_n}{1+n^\rho d_n} = \frac{\delta_n}{1+\frac{1}{n}}$ with $\delta_n = \begin{cases} \frac{1}{n^{1+\rho}}, & n \text{ odd}, \\ \frac{1}{n^\rho}, & n \text{ even}. \end{cases}$

Evidently, $\sum_1^{2n} \delta_r > \sum_1^n \frac{1}{(2r)^\rho}$. Here, the sum on the right is the nth partial sum of the divergent series $\sum_1^\infty \frac{1}{(2n)^\rho}$ $(\rho \leqq 1)$. Hence $\sum \delta_n$ is also divergent, and our series diverges with $\sum \delta_n$.

(2) Let $d_n = \begin{cases} \dfrac{1}{n^{1+\rho}}, & n \neq 2^p, \\ n^{1-\rho}, & n = 2^p. \end{cases} \quad (p \geqq 1)$

As before, this comes under case (c).

We have $\dfrac{d_n}{1+n^{\rho} d_n} = \dfrac{\delta_n}{1+\dfrac{1}{n}}$ with $\delta_n = \begin{cases} \dfrac{1}{n^{1+\rho}}, & n \neq 2^p \\ \dfrac{1}{n^{\rho}}, & n = 2^p \end{cases} \quad (p \geqq 1).$

Evidently, $\sum_{1}^{n} \delta_r < \sum_{1}^{n} \dfrac{1}{r^{1+\rho}} + \sum_{1}^{n} \dfrac{1}{(2^r)^{\rho}}$. Here, the two sums on the right are respectively the nth partial sums of the two convergent series $\sum_{1}^{\infty} \dfrac{1}{n^{1+\rho}}, \sum_{1}^{\infty} \left(\dfrac{1}{2^{\rho}}\right)^n$ $(\rho > 0)$. Hence $\sum \delta_n$ is also convergent, and our series converges with $\sum \delta_n$.

(B) BY G. SANKARANARAYANAN, *Annamalai University.*

Problem.

It is given that

$$V_n = U_n + kU_{n+1} \text{ is convergent.}$$

Show that U_n converges if $|k| > 1$. Show that this will not in general be true when $|k| \leqslant 1$.

Proof.

$$V_n = U_n + kU_{n+1}$$

$$V_1 = U_1 + kU_2$$

$$U_2 = \frac{1}{k}(V_1 - U_1)$$

$$U_3 = \frac{1}{k}(V_2 - U_2)$$

$$= \frac{1}{k}\left[V_2 - \frac{1}{k}(V_1 - U_1)\right]$$

$$= \frac{1}{k}V_2 - \frac{1}{k^2}(V_1 - U_1).$$

In general

$$U_n = \frac{1}{k}V_{n-1} - \frac{1}{k^2}V_{n-2} + \frac{1}{k^3}V_{n-3} - \frac{1}{k^4}V_{n-4} + \cdots + (-1)^n \frac{1}{k^{n-1}}(V_1 - U_1).$$

Let $V_n = l + \varepsilon_n$ where $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Given ε, $|\varepsilon_n| < \varepsilon$ for $n \geqslant n_0$.

$$\therefore \quad U_n = l\left[\frac{1}{k} - \frac{1}{k^2} + \cdots + \frac{(-1)^n}{k^{n-1}}\right] + \left[\frac{\varepsilon_{n-1}}{k} - \frac{\varepsilon_{n-2}}{k^2} + \cdots\right] = T_n + X_n.$$

When $k > 1$, $T_n \longrightarrow \dfrac{l}{k+1}$ as $n \longrightarrow \infty$

and $$|X_n| \leqslant \varepsilon\left[\frac{1}{|k|} + \frac{1}{|k|^2} + \cdots + \frac{1}{|k|^{n-n_0}}\right] + \frac{1}{|k|^{n-n_0+1}}\left[1 + \frac{1}{|k|} + \cdots\right] H$$

where H is the upper bound of ε_{n_0-1}, ε_{n_0-2}, etc.

$$\therefore \quad |X_n| \leqslant \frac{\varepsilon}{|k|-1} + \varepsilon \text{ if } n \geqslant N \text{ (say).}$$

$$\leqslant \varepsilon\,\frac{|k|}{|k|-1} \text{ if } n > N.$$

So U_n converges if $|k| > 1$.

Note.

This will not in general be true when $|k| \leqslant 1$.

A few examples are given below to illustrate this.

(i) Let $k = 1$, $U_n = (-1)^n$

$\therefore \quad V_n = (-1)^n + (-1)^{n+1} = 0$

$\therefore \quad V_n$ converges though U_n does not.

(ii) Let $k = -1$, $U_n = 1 + 1/2 + 1/3 + \cdots + 1/n$

$$V_n = U_n - U_{n+1} = -\frac{1}{n+1}$$

$\therefore \quad V_n \longrightarrow 0$ as $n \longrightarrow \infty$

Hence V_n converges though U_n diverges.

(iii) Let $k < 1$, $U_n = \left(-\dfrac{1}{k}\right)^n$

$$\therefore \quad V_n = \left(-\frac{1}{k}\right)^n + k\left(-\frac{1}{k}\right)^{n+1} = 0.$$

So V_n converges though U_n diverges as $n \longrightarrow \infty$

A more generalized statement of the above problem :

If $V_n = U_n + k_n U_{n+1}$ and if V_n converges, U_n converges if $|k_n| \geqslant 1 + \delta$ where $\delta > 0$.

Statical note on certain Algebraic Inequalities

BY

HARI DAS BAGCHI AND KANTISH CHANDRA MAITY

Calcutta.

1. Suppose that $(a_1, a_2, \ldots, a_n)$ and $(m_1, m_2, \ldots, m_n)$ are two sets of real quantities, such that:

$$0 < a_1 < a_2 < \ldots < a_n \tag{1}$$

and
$$0 < m_1 < m_2 < \ldots < m_n. \tag{2}$$

Any point O on a line l being chosen as origin, we cut off lengths OA_1, OA_2, .., OA_n equal respectively to $a_1, a_2, \ldots, a_n$.

Firstly, we place n equal particles respectively at the n points $A_1, A_2, \ldots, A_n$ and call their mass-centre as G. Then we have:

$$OG = \frac{\sum_{r=1}^{n} a_r}{n}. \tag{3}$$

Secondly, we place particles of masses $m_1, m_2, \ldots\ldots, m_n$ respectively at the n points $A_1, A_2, \ldots, A_n$ and call the mass-centre as G_1. Then

$$OG_1 = \frac{\sum_{r=1}^{n} m_r a_r}{\sum_{r=1}^{n} m_r}. \tag{4}$$

On account of (2), it is clear from statical considerations that the new mass-centre G_1 will lie to the right of the original mass-centre G;

i.e.,
$$OG < OG_1,$$

i.e.,
$$\frac{\sum_{r=1}^{n} a_r}{n} < \frac{\sum_{r=1}^{n} m_r a_r}{\sum_{r=1}^{n} m_r}. \tag{5}$$

Thirdly, we place particles of masses $m_1 a_1$, $m_2 a_2, \ldots, m_n a_n$ respectively at the same set of points $A_1, A_2, \ldots, A_n$. Then if G_2 be the corresponding mass-centre, we have

$$OG_2 = \frac{\sum_{r=1}^{n} m_r a_r^2}{\sum_{r=1}^{n} m_r a_r}. \tag{6}$$

Comparing the mass-distributions in the second and third cases, we notice that in the latter case the masses preponderate *comparatively to the right.* Consequently we get:

$$OG_2 > OG_1,$$

$$\text{i.e.,}\quad \frac{\sum_{r=1}^{n} m_r a_r}{\sum_{r=1}^{n} m_r} < \frac{\sum_{r=1}^{n} m_r a_r^2}{\sum_{r=1}^{n} m_r a_r}. \qquad (7)$$

Proceeding in this way, we have

$$\frac{\sum_{r=1}^{n} a_r}{n} < \frac{\sum_{r=1}^{n} m_r a_r}{\sum_{r=1}^{n} m_r} < \frac{\sum_{r=1}^{n} m_r a_r^2}{\sum_{r=1}^{n} m_r a_r} < \frac{\sum_{r=1}^{n} m_r a_r^3}{\sum_{r=1}^{n} m_r a_r^2} < \cdots < \frac{\sum_{r=1}^{n} m_r a_r^{\nu}}{\sum_{r=1}^{n} m_r a_r^{\nu-1}} < \cdots \qquad (8)$$

Multiplication of two or more of these inequalities manifestly gives rise to any number of subsidiary inequalities.

As an instance, the inequality.

$$\left(\frac{\sum_{r=1}^{n} a_r}{n}\right)^{\nu} < \frac{\sum_{r=1}^{n} m_r a_r^{\nu}}{\sum_{r=1}^{n} m_r}. \qquad (9)$$

holds for every positive integer ν.

2. Evidently the two finite sums:

$$\sum_{r=1}^{n} a_r \quad \text{and} \quad \sum_{r=1}^{n} m_r a_r^{\nu}$$

remain *invariant*, when the constituent terms are permuted among themselves in the most *arbitrary* manner. Consequently the inequalities of Art. 1 must hold good, if (1) and (2) be changed simultaneously to:

$$a_1 > a_2 > \ldots > a_n > 0 \qquad (1')$$

and

$$m_1 > m_2 > \ldots > m_n > 0. \qquad (2')$$

The same result follows also from statical considerations. Following the method of Art. 1, represent the sequence of positive quantities $\{a_n\}$ by the sequence of real points $\{A_n\}$ on a right line l.

If now G be the mass-centre of equal particles, placed at $(A_1, A_2, \ldots, A_n)$ and G_1 be the mass-centre of $(m_1, m_2, \ldots, m_n)$ placed respectively at $(A_1, A_2, \ldots, A_n)$, the relations (3) and (4) of Art. 1 will stand. Further, in view of the relations (2′), we observe that here also the masses in the *second* case preponderate on the *right*

side of the line in comparison with the masses in the *first* case. So G_1 will lie to the right of G. In other words, the relation (5) will still be valid. For a similar reason, the other inequalities of Art. 1 will hold *in toto.*

We may accordingly present the amalgamated result in the following form :

If $(a_1, a_2, \ldots, a_n)$ *and* $(m_1, m_2, \ldots, m_n)$ *be two sets of positive quantities and if both of them are in ascending or descending order and if* u_ν *be defined by* :

$$u_\nu = \frac{\sum_{r=1}^{n} m_r a_r^{\nu}}{\sum_{r=1}^{n} m_r a_r^{-1}}.$$

then $\{u_\nu\}$ *is a monotonically increasing set.*

On the zeros of a certain Complex Polynomial *

BY

C. S. VENKATARAMAN, *Trivandrum*

1. In a note in Vol. VI, 1938, p. 77 of the *Math. Student,* the late Prof. K. S. K. Iyengar showed that the zeros of

$$\left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}\right)$$

lie in $|z| < n$ for all values of n. The object of the present paper is to establish by a similar method, the more general result that all the zeros of $f(z) \equiv \frac{1}{r!} + \frac{z}{(r+1)!} + \frac{z^2}{(r+2)!} + \cdots + \frac{z^n}{(r+n)!}$ lie in the region $(r+1) \leqslant |z| \leqslant (n+r)$.

2. First, we prove that the zeros for all values of n lie in $|z| \geqslant (r+1)$.

Writing $a_k = \frac{1}{(r+k)!}$, we have $f(z) = \sum_{k=0}^{n} a_k z^k$, and

$$(r+1-z)\, f(z) = (r+1)\, a_0 - \left\{ \sum_{k=0}^{(n-1)} \left[a_k - (r+1)\, a_{k+1} \right] z^{k+1} + a_n z^{n+1} \right\}. \qquad (2{\cdot}1)$$

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* I am indebted to Dr. V. Ganapathy Iyer, for his help in connection with this paper.

Now $a_k-(r+1)\,a_{k+1} = \dfrac{1}{(r+k)!}-\dfrac{r+1}{(r+k+1)!}=\dfrac{k}{(r+k+1)!}\geqslant 0$ for $k=0,1,2,\cdots$

Hence if $|z|<(r+1)$,

$$\left|\sum_{k=0}^{(n-1)}\left[a_k-(r+1)a_{k+1}\right]z^{k+1}+a_nz^{n+1}\right|$$

$$\leqslant \sum_{k=0}^{(n-1)}\left[a_k-(r+1)\,a_{k+1}\right]|z|^{k+1}+a_n|z|^{n+1}$$

$$<\sum_{k=0}^{(n-1)}\left[a_k-(r+1)a_{k+1}\right](r+1)^{k+1}+a_n\,(r+1)^{n+1}$$

$$<(r+1)\,a_0. \tag{2·2}$$

Therefore from (2·1)

$$\left|(r+1-z)\,f(z)\right| \geqslant \left|(r+1)\,a_0\right| - \left|\sum_{k=0}^{(n-1)}\left[a_k-(r+1)\,a_{k+1}\right]z^{k+1}+a_nz^{n+1}\right|$$

$$>(r+1)\,a_0-(r+1)\,a_0 \text{ using (2·2)}$$

$$=0.$$

Hence $|f(z)|>0$ if $|z|<(r+1)$; that is, all the zeros of $f(z)$ lie in the region $|z|\geqslant(r+1)$.

3. We next show that if z_o is any zero of $f(z)$, then $|z_o|\leqslant(n+r)$.

Now
$$f\left(\frac{n+r}{z}\right)\equiv\frac{1}{r!}+\left(\frac{n+r}{z}\right)\frac{1}{(r+1)!}+\cdots+\left(\frac{n+r}{z}\right)^n\frac{1}{(r+n)!}. \tag{3·1}$$

Let
$$g(z)\equiv z^nf\left(\frac{n+r}{z}\right)$$

$$\equiv\frac{z^n}{r!}+\frac{(n+r)}{(r+1)!}z^{n-1}+\cdots+\frac{(n+r)^n}{(r+n)!}$$

$$\equiv b_oz^n+b_1z^{n-1}+\cdots+b_n$$

where $b_k=\dfrac{(n+r)^k}{(r+k)!}$.

It is clear that $z=\dfrac{n+r}{z_o}$ is a zero of (3·1) and therefore of $g(z)$ also.

We have $(1-z)\,g(z)=b_n-\{(b_n-b_{n-1})z+\cdots+(b_1-b_o)\,z^n+b_oz^{n+1}\}$.

Now,

$$b_k-b_{k-1}=\frac{(n+r)^k}{(r+k)!}-\frac{(n+r)^{k-1}}{(r+k-1)!}=\frac{(n+r)^{k-1}}{(r+k-1)!}\left[\frac{n+r}{r+k}-1\right]$$

$$\equiv\frac{(n+r)^{k-1}}{(r+k-1)!}\left[\frac{n-k}{r+k}\right]\geqslant 0,\ \text{for } k=1,2,\cdots,n,$$

Hence if $|z|<1$, it can be seen, similarly as in the proof of the first part, that $|(1-z)\,g(z)|>0$.

Therefore $g(z)>0$ if $|z|<1$.

Hence $\left|\dfrac{n+r}{z_0}\right| \geqslant 1$, i.e. $|z_0| \leqslant (n+r)$.

Thus all the zeros of $f(z)$ lie in the region $(r+1) \leqslant |z| \leqslant (n+r)$.

It is obvious that when $r=0$, we get Prof. K. S. K. Iyengar's result which has to be modified as follows :—*For all values of* n, *all the zeros of* $\left(1+z+\cdots+\dfrac{z^n}{n!}\right)$ *lie in the region* $1 \leqslant |z| \leqslant n$.

A Theorem on Congruence

BY

C. S. VENKATARAMAN, *Trichur*

1. In the *Math. Student*, Vol. IX, No. 2, (1941), p. 93, the following question was proposed by M. S. Srinivasan and solved by Dr. Hansraj Gupta and others including the author of the present note :—

(Q. 1780). If p be prime and $M=1\cdot3\cdot5\cdots(p-2)$, prove that $M^4 \equiv 1 \pmod{p}$. (1·1)

In the *Math. Student*, vol. XI, (1943), p. 49, I generalised the above problem thus :

If m is any positive integer and $M=\prod_{r=1}^{k} m_r$ where $m_r = r$th odd number which is less than m and prime to it and $k=$ the number of such m_r's, then $M^4 \equiv 1 \pmod{m}$ (1·2)

The object of the present note is to give the following theorem which is a supplement to (1·2) and to deduce the result corresponding to (1·1).

2. Theorem. *If m is any positive odd integer and $N=\prod_{r=1}^{k} n_r$ where $n_r = r^{th}$ even number which is less than and prime to m and k is the number of such n_r's, then $N^4 \equiv 1 \pmod{m}$.*

(It is obvious that there is no similar theorem when m is *even* since in that case no *even* number is prime to m).

The proof is on the same lines as that of (1·2).

Proof. Let $m - n_r = m_r$. Then m_r is odd, less than and prime to m. Also the number of m_r's is exactly k.

$$\text{Hence, } N = n_1 \cdot n_2 \cdot \cdots \cdot n_k$$
$$\equiv (-1)^k\, m_1 \cdot m_2 \cdot \cdots \cdot m_k \pmod{m}.$$

$\therefore$ $N^2 \equiv (-1)^k P \pmod{m}$ where P is the product of all numbers less than and prime to m.

$\therefore$ $N^4 \equiv 1 \pmod{m}$ by Gauss's extension of Wilson's theorem.

Corollary. If p is an odd prime, and $N = 2 \cdot 4 \cdot 6 \cdots (p-1)$, then $N^4 \equiv 1 \pmod{p}$.

A Note on Generalised Mean Value Theorem

BY

B. N. MUKHERJEE, *Scottish Church College, Calcutta*

The Mean-value theorem of Differential Calculus has been generalised in various ways by Peano, Mahajani, Rajgopal, Ganapathy Iyer, Thiruvenkatachar and others. The object of this note is to give a generalised form of the Mean-value theorem.

$$\text{Let } \phi(x) = \begin{vmatrix} 1 & F(b) - F(a) & F(x) \\ 1 & G(b) - G(a) & G(x) \\ 1 & H(b) - H(a) & H(x) \end{vmatrix}$$

where $F(x)$, $G(x)$, $H(x)$ are continuous in $a \leqslant x \leqslant b$ and differentiable in $a < x < b$.

Then $\phi(a) = \phi(b)$; hence by Rolle's theorem $\phi'(\xi) = 0$, $a < \xi < b$.

$$\therefore \quad \begin{vmatrix} 1 & F(b) - F(a) & F'(\xi) \\ 1 & G(b) - G(a) & G'(\xi) \\ 1 & H(b) - H(a) & H'(\xi) \end{vmatrix} = 0. \tag{1·1}$$

$$\left.\begin{aligned} \text{Let } F(x) &= f(x) + \frac{b-x}{1!} f'(x) + \frac{(b-x)^2}{2!} f''(x) + \cdots + \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x) \\ G(x) &= g(x) + \frac{b-x}{1!} g'(x) + \frac{(b-x)^2}{2!} g''(x) + \cdots + \frac{(b-x)^{p-1}}{(p-1)!} g^{p-1}(x) \\ H(x) &= h(x) + \frac{b-x}{1!} h'(x) + \frac{(b-x)^2}{2!} h''(x) + \cdots + \frac{(b-x)^{q-1}}{(q-1)!} h^{q-1}(x) \end{aligned}\right\} \tag{1·2}$$

Then $F(b) - F(a) = f(b) - f(a) - (b-a) f'(a) - \cdots - \dfrac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$ and let us denote this by $R_n(f)$.

$$F'(x) = \frac{(b-x)^{n-1}}{(n-1)!} f^n(x).$$

$\therefore$ from (1·1) we get

$$\begin{vmatrix} 1 & R_n(f) & \frac{(b-\xi)^{n-1}}{(n-1)!} f^n(\xi) \\ 1 & R_p(g) & \frac{(b-\xi)^{p-1}}{(p-1)!} g^p(\xi) \\ 1 & R_q(h) & \frac{(b-\xi)^{q-1}}{(q-1)!} h^q(\xi) \end{vmatrix} = 0. \qquad (1\cdot3)$$

This is the required generalised form.

Conditions for continuity and differentiability of $F(x)$, $G(x)$, $H(x)$ imply that $f^{n-1}(x)$, $g^{p-1}(x)$, $h^{q-1}(x)$ are continuous in $a \leqslant x \leqslant b$ and $f^n(x)$, $g^p(x)$, $h^q(x)$ exist in $a < x < b$.

This form is similar to that given by Rajgopal.[1]

By choosing properly the functions $G(x)$, $H(x)$ in (1·2), various forms of the Mean value theorem can be obtained. Thus when $H(x) = c$, (a constant), we have

$$\begin{vmatrix} R_n(f) & \frac{(b-\xi)^{n-1}}{(n-1)!} f^n(\xi) \\ R_p(g) & \frac{(b-\xi)^{p-1}}{(p-1)!} g^p(\xi) \end{vmatrix} = 0.$$

This gives Mahajani's form[2].

2. Let us take[3]

$$F(x) = \sum_{r=0}^{n} (-1)^{n-r} U_n^{(n-r)}(b-x) f^{(r)}(x), \text{ where } U_n(b-x) = \sum_{k=0}^{n} a_k \frac{(b-x)^k}{k!}.$$

$$G(x) = \sum_{r=0}^{p} (-1)^{p-r} V_p^{(p-r)}(b-x) g^{(r)}(x), \text{ where } V_p(b-x) = \sum_{k=0}^{p} b_k \frac{(b-x)^k}{k!}$$

$$H(x) = \sum_{r=0}^{q} (-1)^{q-r} W_q^{(q-r)}(b-x) h^{(r)}(x) \text{ where } W_q(b-x) = \sum_{k=0}^{q} c_k \frac{(b-x)^k}{k!}.$$

$$\therefore \quad F(b) - F(a) = \sum_{r=0}^{n} [a_r f^{(n-r)}(b) - (-1)^{n-r} U_n^{(n-r)}(b-a) f^{(r)}(a)]$$

$$F'(x) = U_n(b-x) f^{n+1}(x).$$

Hence substituting in (1·1), we get

$$\begin{vmatrix} 1 & \sum_{r=0}^{n} [a_r f^{(n-r)}(b) - (-1)^{n-r} U_n^{(n-r)}(b-a) f^{(r)}(a)] & U_n(b-\xi) f^{n+1}(\xi) \\ 1 & \sum_{r=0}^{p} [b_r g^{(p-r)}(b) - (-1)^{p-r} V_p^{(p-r)}(b-a) g^{(r)}(a)] & V_p(b-\xi) g^{p+1}(\xi) \\ 1 & \sum_{r=0}^{q} [c_r h^{(q-r)}(b) - (-1)^{q-r} W_q^{(q-r)}(b-a) h^{(r)}(a)] & W_q(b-\xi) h^{q+1}(\xi) \end{vmatrix} = 0$$

which is a generalised form of (1·3); for, (1·3) can be easily deduced from this by taking

$$U_n(b-x)=\frac{(b-x)^n}{n!};\ V_p(b-x)=\frac{(b-x)^p}{p!};\ W_q(b-x)=\frac{(b-x)^q}{q!}.$$

3. Incidentally it may be mentioned that the formula (1·1) is completely equivalent to Cauchy's Mean Value theorem.* Thus subtracting the 1st row from the others, we get

$$\begin{vmatrix} 1 & F(b)-F(a) & F'(\xi) \\ 0 & [G(b)-F(b)]-[G(a)-F(a)] & G'(\xi)-F'(\xi) \\ 0 & [H(b)-F(b)]-[H(a)-F(a)] & H'(\xi)-F'(\xi) \end{vmatrix}=0, \qquad (3\cdot1)$$

If we write $\phi(x)=G(x)-F(x)$, $\psi(x)=H(x)-F(x)$, this becomes

$$\begin{vmatrix} 1 & F(b)-F(a) & F'(\xi) \\ 0 & \phi(b)-\phi(a) & \phi'(\xi) \\ 0 & \psi(b)-\psi(a) & \psi'(\xi) \end{vmatrix}=0. \qquad (3\cdot2)$$

which is equivalent to Cauchy's formula

$$\begin{vmatrix} \phi(b)-\phi(a) & \phi'(\xi) \\ \psi(b)-\psi(a) & \psi'(\xi) \end{vmatrix}=0, \qquad (3\cdot3)$$

Conversely starting from (3·3), we have

$$\begin{vmatrix} 1 & \lambda & \mu \\ 0 & \phi(b)-\phi(a) & \phi'(\xi) \\ 0 & \psi(b)-\psi(a) & \psi'(\xi) \end{vmatrix}=0. \qquad (3\cdot4)$$

for arbitrary λ and μ.

If in this result we write $\lambda=F(b)-F(a)$, $\mu=F'(\xi)$, $\phi(x)=G(x)-H(x)$ and $\psi(x)=H(x)-F(x)$, we obtain (3·1), and (1·1) follows thence by adding the first row to the others.

4. It should be noted that (1·5) can also be obtained in a slightly different manner. Let

$$\phi(x)=\begin{vmatrix} 1 & F(a) & F(x) \\ 1 & G(a) & G(x) \\ 1 & H(a) & H(x) \end{vmatrix} \qquad (4\cdot1)$$

where $F(x)=f(b)-\sum_{r=0}^{n-1}\frac{(b-x)^r}{r!}f^{(r)}(x)$

$$G(x)=g(b)-\sum_{r=0}^{p-1}\frac{(b-x)^r}{r!}g^{(r)}(x)$$

$$H(x)=h(b)-\sum_{r=0}^{q-1}\frac{(b-x)^r}{r!}h^{(r)}(x)$$

* Suggested by Sri. V. R. Thiruvenkatachar.

and suppose that $f^{n-1}(x)$, $g^{p-1}(x)$, $h^{q-1}(x)$ are continuous in $a \leqslant x \leqslant b$ and $f^{n}(x)$, $g^{p}(x)$, $h^{q}(x)$ exist in $a < x < b$. Then since $\phi(a) = \phi(b) = 0$, hence $\phi'(\xi) = 0$, $a < \xi < b$.

Since $F(a) = R_n(f)$ and $F'(x) = -\dfrac{(b-x)^n}{(n-1)!} f^n(x)$, we get the relation (1·1).

The author wishes to express his thanks to Prof C. N. Srinivasiengar for his keen interest in the paper.

References

1. Rajgopal C. T. *Mathematics Student* Vol. 14 (1946), 71–73

2. Mahajani G. S. *Messenger of Mathematics* Vol. 52, pp. 78–80.

3. Mahajani and Thiruvenkatachar: *Proc. Ind. Acad. Sc.* Vol. XXI, 1950, 124–9

[The form (1·1) of the mean value theorem is interesting and worth noting. Taking $F(x) = \dfrac{f(x)}{f(a)}$, etc., we deduce the familiar form

$$\begin{vmatrix} f(a) & f(b) & f'(\xi) \\ g(a) & g(b) & g'(\xi) \\ h(a) & h(b) & h'(\xi) \end{vmatrix} = 0. \qquad (1\cdot 11)$$

But (1·11) will not lead to (1·1). The first column in (1·1) can be replaced by any constants l, m, n.—*Ed.*].

A property of the cubical parabola

BY

B. K. SEN GUPTA, *Srikhanda*

Any point P on the cubical parabola $y = x^3$ can be taken as (θ, θ^3). Transferring the orgin to P, the equation of the curve becomes

$$y = 3\theta^2 x + 3\theta x^2 + x^3.$$

The equation to the osculatiṇg conic at P is found to be

$$45\theta^4 x^2 + 15\theta^2 xy - y^2 + 27\theta^3(3\theta^2 x - y) = 0.$$

This meets the curve again at Q $(-6\theta, -1260^3)$, and the equation to the common chord is $21\theta^2 x - y = 0$, which when referred to the old origin becomes $21\theta^2 x - y - 20\ \theta^3 = 0$. The envelope of this line is $100y = 343x^3$, a curve of the same species.

PQ meets the curve again in the point R whose coordinates referred to P are $(3\theta, 63\theta^3)$. It follows that PQ = 2PR.

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-600 008

The envelope of the common chord of the cubical parabola $y=x^3$ and the osculating conic to it at any point is another cubical parabola, and the entire intercept on the common chord made by the cubic is trisected at the point of contact.

Similarly. for the semi-cubical parabola $y^3=x^2$, the envelope of the common chord between the curve and the osculating conic at any point is found to be another semi-cubical parabola whose equation is

$$29791y^3=6075x^2.$$

Lecture Notes

BY

B. B. BAGI. *Dharwar.*

1. Directrices of a conic

The equation of the directrices of the conic

$$S(x,y,z)\equiv Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$$

is given by

$$4D\Delta^2S+\phi(1,1,1)r^2[\sum bc\cos A\,(\eta-\zeta)^2=0 \qquad (1)$$

where D is the discriminant, $\phi(l,m,n)=0$ the tangential equation, r the length of the semi-axis of the conic and Δ the area of the triangle of reference, the coordinates being areal.

None of the current text books on the subject including Salmon's gives the above result. Smith makes a 'vague reference to the method used in the case of Cartesian coordinates. The method has to be modified before it can be extended to homogeneous coordinates. The modified proof is indicated below :

We observe that the directrices are a pair of parallel straight lines passing through the common points of the conic and its director circle.

Now the equation of the director circle is

$$(pbc\cos A+qca\cos B+rab\cos C)S$$
$$=bc\cos A(\eta-\zeta)^2+ca\cos B(\zeta-\xi)^2+ab\cos C(\xi-\eta)^2$$

[§ 189, p. 293, Askwith].

Hence the general equation of conics passing through the common points of $S=0$ and its director circle is of the form

$$\lambda S+[bc\cos A(\eta-\zeta)^2+ca\cos B(\zeta-\xi)^2+ab\cos C(\xi-\eta)^2]=0. \qquad (2)$$

This can be transformed into

$$\lambda\phi(\xi,\eta,\zeta)+D[\textstyle\sum bc\cos A(\eta-\zeta)^2]=0. \qquad (3)$$

Now the left hand side of (3) can be broken up into two linear factors if

$$\begin{vmatrix} \lambda A'+a^2D & \lambda H'-ab\cos C\cdot D & \lambda G'-ca\cos B\cdot D \\ \lambda H'-ab\cos C\cdot D & \lambda B'+b^2D & \lambda F'-bc\cos A\cdot D \\ \lambda G'-ac\cos B\cdot D & \lambda F'-bc\cos A\cdot D & \lambda C'+c^2D \end{vmatrix}=0. \qquad (4)$$

This equation reduces to

$$\lambda^3D^2+\lambda^2D^2[\textstyle\sum a^2A-2\sum bc\cos A\cdot F]+4\lambda D^2\Delta^2\phi(1,1,1)=0.$$

Removing the factor D^2 and the factor λ which corresponds to the pair of diameters

$$\textstyle\sum bc\cos A(\eta-\zeta)^2=0$$

we get the quadratic in λ

$$\lambda^2+\lambda[\textstyle\sum a^2A-2\sum bc\cos A\cdot F]+4\Delta^2\phi(1,1,1)=0, \qquad (5)$$

giving two values of λ corresponding to each of which we have a pair of parallel directrices.

To complete the proof we further observe that the lengths of axes of $S=0$ are given by

$$\begin{vmatrix} A'+kr^2a^2 & H'-kr^2ab\cos C & G'-kr^2ac\cos B \\ H'-kr^2ab\cos C & B'+kr^2b^2 & F'-kr^2bc\cos A \\ G'-kr^2ac\cos B & F'-kr^2bc\cos A & C'+kr^2c^2 \end{vmatrix}=0 \qquad (6)$$

where $k=\phi\ (1,\ 1,\ 1)/4\Delta^2$.

Now it is easy to see that (4) changes into (6) when D/kr^2 is written for λ in (4). Hence writing D/kr^2 for λ in (3) above, we get the equation of a pair of directrices in the required form.

2. Tangents to a conic at the extremities of a chord

To find the equation of the pair of tangents drawn at the extremities of the chord $lx+my+nz=0$ to the conic $S(x,y,z)=0$.

We observe that if $(x_1,\ y_1,\ z_1)$ is a point on either of the two tangents, the polar of $(x_1,\ y_1,\ z_1)$ passes through the point of contact of that tangent.

Now the point of contact is the point of intersection of

$$lx+my+nz=0 \text{ and } x\xi_1+y\eta_1+z\zeta_1=0.$$

Hence its coordinates are proportional to

$$n\eta_1-m\zeta_1,\ l\zeta_1-n\xi_1,\ \xi_1m-\eta_1l.$$

This point lies on $S(x,y,z)=0$.

$$\therefore \quad S[n\eta_1 - m\zeta_1,\ l\zeta_1 - n\xi_1,\ \xi_1 m - \eta_1 l] = 0.$$

When the suffixes are suppressed, we get the desired equation in the form $S[n\eta - m\zeta,\ l\zeta - n\xi,\ m\xi - l\eta] = 0$.

By writing $l=m=n=1$ in this, we get the equation of the aspmytotes $S[\eta - \zeta,\ \zeta - \xi,\ \xi - \eta] = 0$.

A Correction

BY

S. SURYANARAYANA IYER, *St. Joseph's College, Trichy*

The Corollary 1 on page 76 of Mr. C. T. Rajagopal's article on 'Reciprocal screws' in the *Mathematics Student* Vol. XIV (3 & 4) is incorrect.

It says—"*Given* a tetrahedron ABCD, it is necessary and sufficient for two wrenches F and F′ to act on reciprocal screws, that AB, CD should be conjugate lines of F and AC, BD should be conjugate lines of F′."

The wording is not clear. When two wrenches F and F′ are given and the tetrahedron ABCD is given, it may not be possible to replace F by two forces, one along AB and the other along CD. Even supposing that F is replaceable by P and Q along AB, CD, F′ need not be replaceable by two forces P′, Q′ acting along AC, BD.

For, replacing F′ by a force P′ along AC and another force Q′, we see that Q′ is a nul line of F i.e., of the system P, Q; but it need not intersect P and Q. For example, Q′ may be any line intersecting at right angles the wrench F without intersecting P, Q i.e. AB, CD.

Of course, the condition is sufficient, vig., If F and F′ are replaceable by P, Q along AB, CD and P′, Q′ along AC, BD, then F and F′ will be reciprocal.

NOTE BY THE EDITOR

Mr. L. C. Hsu points out that Mr. Chakrabarti's identity

$$\binom{x}{1}_n = \sum_{0\leq p\leq n} \binom{\delta+n-1}{-1}_p \binom{x-\delta}{-1}_{n-p} \binom{n}{p} \qquad (*)$$

mentioned in the paper by S. C. Chakrabarti: On Kemmer's Identity in Combinatory Analysis, *Math. Student*, Vol. XVIII, 43-45 is actually the classical Van der Monde's identity (See, e.g. Jordan: *Calculus of Finite Differences*, p. 48) which goes back to probably 1722. For, by dividing both sides of (*) by $n\,!$, we may rewrite (*) in the form

$$\binom{x+n-1}{n} = \sum_{0\leq p\leq n} \binom{\delta+n-1}{p} \binom{x-\delta}{n-p}.$$

SOLUTIONS TO QUESTIONS

Question 1801

(P. Kesava Menon). If $f(n) = \cos^{-1} 2^{-n/2}({}_nC_0 - {}_nC_2 + {}_nC_4 - \cdots)$ show that $f(p) + f(q) = f(p+q)$.

Solution and remarks by C. S. Venkataraman

We have

$$\cos n\theta = \cos^n \theta\, [{}_nC_0 - {}_nC_2 \tan^2\theta + {}_nC_4 \tan^4\theta - \cdots\cdots]. \qquad (1)$$

Putting $\theta = \frac{\pi}{4}$ in (1),

$$\cos \frac{n\pi}{4} = \left(\frac{1}{2}\right)^{n/2} [{}_nC_0 - {}_nC_2 + {}_nC_4 - \cdots\cdots].$$

Hence $f(n) = \cos^{-1} 2^{-n/2}({}_nC_0 - {}_nC_2 + {}_nC_4 - \cdots) = \frac{n\pi}{4}$,

taking the principal value.

Therefore $f(p) + f(q) = \frac{p\pi}{4} + \frac{q\pi}{4} = \frac{(p+q)\pi}{4} = f(p+q)$.

Remarks: A similar property is possessed by

$$\psi(n) = \sin^{-1} 2^{-n/2}({}_nC_1 - {}_nC_3 + {}_nC_5 - \cdots\cdots)$$

and also by

$$F(n) = f(n) + \psi(n).$$

Further

$$f(p_1 + p_2 + \cdots\cdots + p_r) = f(p_1) + f(p_2) + \cdots\cdots + f(p_r)$$

and

$$f(p_1 p_2 \ldots\ldots p_r) = \left(\frac{4}{\pi}\right)^{r-1} f(p_1)\, f(p_2) \ldots\ldots f(p_r).$$

Question 1818

(S. Sitharaman). ABC is a plane triangle. With the usual notation, prove that

$$\sin^{-1}\left(\cos\frac{A}{2}\sqrt{1-\frac{a}{s}}\right)+\sin^{-1}\left(\cos\frac{B}{2}\sqrt{1-\frac{b}{s}}\right)+\sin^{-1}\left(\cos\frac{C}{2}\sqrt{1-\frac{c}{s}}\right)=\pi/2$$

Solution 1: by the proposer

2. by S. S. Subrahmanyam, C. S. Venkataraman, V. Venkataramana Rao, M. S. Narasimhan, R. Ramamohana Rao and Vepa Purniah.

1. Let α, β, γ be the angles of the triangle whose sides are $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$. This triangle exists when the original triangle exists, since $\sqrt{b}+\sqrt{c} > \sqrt{a}$, etc.

$$\text{Now} \quad \cos\frac{A}{2}\cdot\sqrt{1-\frac{a}{s}}=\frac{b+c-a}{2\sqrt{bc}}=\cos\alpha.$$

$$\text{Hence} \quad \cos^{-1}\left(\cos\frac{A}{2}\cdot\sqrt{1-\frac{a}{s}}\right)+\cos^{-1}\left(\cos\frac{B}{2}\cdot\sqrt{1-\frac{b}{s}}\right)+\cos^{-1}\left(\cos\frac{C}{2}\cdot\sqrt{1-\frac{c}{s}}\right)=\pi$$

$$\therefore \quad \sin^{-1}\left(\cos\frac{A}{2}\sqrt{1-\frac{a}{s}}\right)+\sin^{-1}\left(\cos\frac{B}{2}\cdot\sqrt{1-\frac{b}{s}}\right)+\sin^{-1}\left(\cos\frac{C}{2}\cdot\sqrt{1-\frac{c}{s}}\right)=\frac{\pi}{2}.$$

2. If $\sin^{-1}\left(\cos\frac{A}{2}\cdot\sqrt{1-\frac{a}{s}}\right)=\alpha$, etc., we have

$$\sin\alpha=\frac{s-a}{\sqrt{bc}}, \quad \cos\alpha=\sqrt{\left(\frac{\sum bc-s^2}{bc}\right)}.$$

$$\text{Hence} \tan\alpha=\frac{s-a}{\sqrt{(\sum bc-s^2)}}$$

$\therefore \quad \sum \tan\beta\tan\gamma=1$, so that $\alpha+\beta+\gamma=\pi/2$.

Or, $\cos(\alpha+\beta)$ is verified to be $\frac{s-c}{\sqrt{ab}}$ i.e. $\sin\gamma=\cos(\alpha+\beta)$

Hence $\alpha+\beta+\gamma=\pi/2$.

Question 1819

(S. Sitharaman). The harmonic polar of the symmedian point w.r.t. a ΔABC meets BC, CA, AB in DEF. If AD, BE, CF are the lines $L=0$, $M=0$, $N=0$, the coordinates being areal, show that the conic

$$\frac{\cot\frac{A}{2}}{L}+\frac{\cot\frac{B}{2}}{M}+\frac{\cot\frac{C}{2}}{N}=0$$

is a Fragnon ellipse if $\cos A + \cos B + \cos C = \sqrt{6} - 1$, and that one focus is at the circumcentre of ΔABC.

[In a Fragnon ellipse, the minor auxiliary circle passes through the real foci; $e = 1/\sqrt{2}$.].

Solution by M. S. Narasimhan and by the proposer.

From the property of the symmedian point, $L=0$, $M=0$, $N=0$ are the tangents at A,B,C to the circumcircle of ABC. The given conic is therefore

$$\frac{\cot \frac{A}{2}}{c^2y + b^2z} + \frac{\cot \frac{B}{2}}{a^2z + c^2x} + \frac{\cot \frac{C}{2}}{b^2x + a^2y} = 0. \tag{1}$$

It will be verified that this is the polar reciprocal of the incircle w.r.t. the circumcircle of ΔABC. One of its foci is therefore at the circumcentre of ΔABC. The eccentricity e is given by

$$e = \frac{\sqrt{R^2 - 2Rr}}{r} \quad \text{i.e.} \quad \frac{R^2}{r^2} - 2\frac{R}{r} - e^2 = 0 \cdot$$

$$\therefore \quad \frac{R}{r} = 1 + \sqrt{(1+e^2)}$$

$$\therefore \quad \frac{r}{R} = \frac{\sqrt{(1+e^2)} - 1}{e^2} = 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} \cdot$$

$$\therefore \quad \cos A + \cos B + \cos C = 1 + 4\frac{r}{R} = \frac{\sqrt{(1+e^2)} - (1-e^2)}{e^2}$$

Hence,

(i) if (1) is a Fragnon ellipse i.e. if $e = 1/\sqrt{2}$, then,
$$\sum \cos A = \sqrt{6} - 1$$

(ii) if (1) is a parabola, $\sum \cos A = \sqrt{2}$

(iii) if (1) is a rectangular hyperbola, $\sum \cos A = \frac{\sqrt{3}+1}{2}$.

QUESTIONS FOR SOLUTION

1821 (V. Chinnathambi). Let N be any number of n digits of which k are zeros, p are a's, q are b's, and so on. Then the sum of all the different numbers of n digits that can be formed by permutations of the given digits is

$$\frac{(n-2)!\,(ap+bq+cr+\cdots)}{9k!\,p!\,q!\,r!}\,\{10^{n-1}\,(10n-10-k)+k+1-n\}.$$

1822 (V. Thébault). Work out the division $abcd = efg \times h + ij$, given that $a+b+c+d = e+f+g+h = i+j$, by employing all the digits 0 to 9.

Example. $1950 = 624 \times 3 + 78$.

1823 (V. Thébault). Determine the fractions $F = \dfrac{a_1a_2a_3 \ldots a_n}{a_na_{n-1} \ldots a_3a_2b_1}$

whose numerator and denominator have each n digits, and which simplify to one or all the fractions

$$\frac{a_1a_2 \ldots a_{n-1}}{a_{n-1}a_{n-2} \ldots a_2b_1},\ \frac{a_1a_2 \ldots a_{n-2}}{a_{n-2}a_{n-3} \ldots a_2b_1},\ \ldots,\ \frac{a_1a_2a_3}{a_3a_2b_1},\ \frac{a_1a_2}{a_2b_1},\ \frac{a_1}{b_1}.$$

Examples. $F = \dfrac{1}{4} = \dfrac{16}{64} = \dfrac{166}{664} = \cdots,\quad F = \dfrac{1}{5} = \dfrac{19}{95} = \dfrac{199}{995} = \cdots,$

$F = \dfrac{2}{5} = \dfrac{26}{65} = \dfrac{266}{665} = \cdots,\quad F = \dfrac{4}{8} = \dfrac{49}{98} = \dfrac{499}{998} = \cdots.$

1824 (V. Thébault). The six radical planes of the spheres described on the orthocentroidal circles of the faces of a tetrahedron as great circles pass through the middle points of the edges.

[The orthocentroidal circle of a triangle is described on the distance between the orthocentre and the centroid.]

1825 (A. Moessner). Obtain the general solution to the equations

$$A+B=C+D;\ A^4+B^4=C^4+D^4.$$

ANNOUNCEMENTS AND NEWS

The Seventeenth Conference of the Indian Mathematical Society will be held in Bangalore in the last week of December 1951, under the auspices of the University of Mysore. The dates and details of arrangements will be intimated to the members in due course. Members intending to present papers to the conference should send brief abstracts to Dr. K. Chandrasekharan, Tata Institute of Fundamental Research, Apollo Pier Road, Bombay 1, before 31st October, 1951.

The following have been admitted as members of the Society; Mr. Tribikram Pati, M.A., Research Student, 122, Ganganath Jha Hostel, Allahabad University, Allahabad.

Mr. Y. V. Thosar, M.Sc., Lecturer, Science College Nagpur.

Dr. J. D. Newberg, Institute for Advanced Study, Princeton N. J., U. S. A.

Mr. V. B. Cholkar, M.Sc., Senior Lecturer, Baroda College.

" Shaik Md. Abdul Wahab, B.Sc., B.E., Lecturer in Civil Engineering, Govt. Engineering College, Karachi.

" G. T. Shah, M.A., Asst. Professor, Pratap College, Amalner (E.K.)

" T. S. Nanjundiah, M.Sc., Senior Research Scholar, Central College, Bangalore.

" S. K. Lakshmana Rao, M.Sc., Lecturer in Applied Math., Indian Institute of Science, Bangalore.

" Som Datt Chopra, M.A., Head of the Department of Maths., Punjab Univ. Cam College, New Delhi.

" Gurbaksh Singh Oberai, M.A., No. 694, Moh. Kaithmajri, Ambala City.

" Ramachandra Ameta, M.A., Head of the Department of Maths., Nalanda College, Biharshrif, Patna.

" P. S. Rau, B.A. (Hons.), Tutor, Andhra University, Waltair.

We regret to report the death of Dr. S. R. U. Savoor, Retd. Director of Public Instruction, Madras and of Prof. N. V. Arunachala Sastry, Retd. Prof. of Maths., Nizam College, Hyderabad. Both were Life Members and had rendered valuable service on the Managing Committee of the Society.

Science Museum Library Overseas Photocopying Service

The Science Museum Library, which is endeavouring to obtain a complete collection of the important scientific and technical periodical literature of the world, has started a photocopying service for organizations outside Great Britain. For this service it is necessary to obtain special requisitions from the Director and Secretary, Science Museum, S. Kensington, London, S. W. 7, England. Interested persons can refer to him for details and for the method of procedure.

Requisitions for copies of publications in the Art Library of the Victoria and Albert Museum should be sent to the Director, Victoria and Albert Museum, S. Kensington, London S. W. 7, England. All other requisitions should be sent to the Science Library.

The Zentralblatt für Mathematik has, in May 1948, resumed publication after an interruption of 3 years, with a review of literature that had appeared since 1947. It reports continuously on the whole mathematical world literature. Copies may be ordered from the publishers, Lange and Springer, Wissenschaft -liche Buchhandlung, Berlin W. 35, West Berlin, Germany.

International Union of Mathematicians

A conference was held in the Halls of Columbia University in New York City from August 27 to 29, 1950, for the purpose of creating a new International Union of Mathematicians. Delegates of National Committees representing the mathematicians of nineteen countries attended. Dr. K. Chandrasekharan and Dr. S. Minakshisundaram were the Indian delegates. (Dr. S. S. Pillai, who was also a delegate, was killed in an air crash while on his way). Transportation grants were given to delegates from out of the funds provided for the purpose by UNESCO.

The convention of delegates framed the Statutes and Bye-laws for the proposed Union, and appointed an Interim Committee to act as its agent in the execution of its directions for the creation of the Union. The Interim Committee is composed of representatives of the National Committees of Denmark, France, Great Britain, India, Italy, Poland and U.S.A. Prof. Borge Jessen (University of Copenhagen) representing Denmark is Secretary of the Interim Committee, and information pertaining to the Union may be obtained from him.

International Congress of Mathematicians

An International Congress of Mathematicians was held in Cambridge, Massachusetts, from August 30 to September, 6, 1950, under the auspices of the American Mathematical Society. Harvard University was the principal host institution.

The Congress was attended by about 1400 mathematicians and 900 guests. Though a vast majority of them belonged to the U.S.A., about 300 members and delegates from other countries were present. Among the Indian mathematicians who attended were the following: Prof. R. C. Bose, Prof. S. Chowla, Dr. K. Chandrasekharan, Dr. Harish-Chandra, Dr. P. C. Mahalanobis, Dr. S. Minakshisundaram, Dr. S. N. Roy, Mr. K. G. Ramanathan, Prof. B. R. Seth.

After the Inaugural Address by Prof. O. Veblen, President of the Congress, and the presentation of the two Fields Medals to Prof. L. Schwartz and Prof. A. Selberg, the work of the Congress proceeded in 4 Conferences (Analysis; Algebra and Theory of Numbers; Topology and Geometry; Applied Mathe-

matics) and 7 Sections (Analysis; Algebra and Theory of Numbers; Topology and Geometry; Probability, Statistics, Actuarial Science and Economics; Mathematical Physics and Applied Mathematics; Logic and Philosophy; History and Education). The Conferences were utilized for presenting fairly detailed reports on recent research in fields in which vigorous advances have just been made or are in progress. The Sectional meetings were meant to provide for the presentation of contributed papers not included in Conference programmes. About 150 papers or reports were read under each of the first four headings. There were, besides, invited hour-addresses by the following mathematicians: A. Beurling, H. Hopf, H. Cartan, R. L. Wilder, S. Bochner, K. Gödel, O. Zariski, A. Weil, M. Morse, H. Whitney, W. V. D. Hodge, J. F. Ritt, H. Davenport, L. Schwartz, A. Wald, and N. Wiener.

Papers were read by the following Indian delegates: Prof. R. C. Bose, Prof. S. Chowla, Dr. K. Chandrasekharan, Dr. S. N. Roy and Prof. B. R. Seth. Dr. K. Chandrasekharan presided over one of the Sessions in Analysis.

REVIEWS

Théorie des Distributions—Tome I

L. Schwartz

Actualités Scientifiques et Industrielles—No. 1091—Hermann et Cie Editeurs 6, Rue de la Sorbonne—Paris, 1950.

As soon as L. Schwartz published his first article on the Theory of Distribution (*), there was only one opinion about it in the world of Mathematics, that this was a great discovery, one which meant not only a considerable advance in a particular field of research but that important progress had been made in many domains having had until then very little in common. For instance, Schwartz himself could apply his theory to difficult questions concerning trigonometrical series and Fourier transforms while, a little later, de Rham could utilize the same to systematize admirably his theory of currents in Algebraic Topology (**).

The volume under review is only the first of two volumes in which Schwartz intends to develop systematically his theory. Naturally the task of giving an account of this theory would have been much easier had the second volume been available, not only because it is very difficult to arrive at any definite opinion about important applications which, we are told will be found in the second volume, but also because the many and often precious references

(*) *Annales de l'Université de Grenoble*, 1945, pp. 57–74.

(**) Mimeographed lectures by G. de Rham and K. Kodaira, Institute for Advanced Study, Princeton, 1950.

which it would be so useful to consult in order to clear a few doubts will all be given only at the end of this treatise. However it is possible to acquire a very good grasp of the notion of distribution and of most of its fundamental properties through a careful study of this first volume.

Schwartz, in his article of 1945, was already defining a distribution as a generalization of the notion of function. Such a generalization had been implicitly used by Dirac in his *Quantum Mechanics*, but without the rigour which mathematics requires. His celebrated "improper function" $\delta(x)$ could be derived and integrated although it was hard to guess why mere manipulations of entities insufficiently defined could ever be so successful. Similarly the *Symbolic Calculus* of Heaviside could be justified only by extremely artificial methods.

The distributions generalize the notion of a function so as to edify the theory of Dirac and the artifices of Heaviside upon the stablest of foundations and such a result is obtained by means, the simplicity of which is really amazing. One cannot but wonder why first class mathematicians have been able to bypass for such a long time the theory of Distributions!

In the present state of development of mathematics, it may be thought that any generalization of the notion of function is to be obtained by defining extensions of certain vector spaces either by classical topologies or by dual (or conjugate) spaces of known functions. This last method seems preferable since such dual spaces are not generally function spaces (as the celebrated theorem of F. Riesz shows) but vector spaces containing a function space as a subspace. Moreover it is a well-known fact that the "poorer" (if one may say so) the function space, all the "richer" his dual is—richer in elements which a skilful topologization may turn into natural generalizations of the "classical" notion of function.

Such is the simple idea on which the theory of distributions is based. Schwartz considers a function space D, whose elements are complex functions of n real arguments ; these functions are indefinitely derivable and vanish, along with all their derivatives, outside of a compact subset of D, the carrier, which generally varies with the function considered. D is not easily topologized. In fact two topologies may be determined in it : one, the generalization of the convergence topology of Fréchet obtained by substituting the notion of filter for that of convergent sequence ; the other, the Hausdorff topology. They are not equivalent In any case, there is no norm, in the sense of Banach. But the dual D′ of D may be, all the same, topologized ; this being done by utilizing the notion of bounded subset in the vector space D. In this manner, a topological vector space is obtained whose elements are called by Schwartz *distributions*. The existence of derivatives and of primitives of these distributions is easily established : it is an immediate consequence of the existence of a topology in D′. A very remarkable fact is that the topology of D′ may be determined independently of the topology introduced in D.

It is obvious that the derivative of a function differentiable in the classical sense is the same as its derivative as a distribution. But a function may not be differentiable in the sense of analysis; yet it has always a derivative in the sense of the theory of Distributions. This is, for instance, the case of Heaviside's function whose derivative is precisely $\delta(x)$. This is also the case of the function r^{-1}, where r stands for the euclidean distance in a three dimensional euclidean space. If Δ denotes the ordinary laplacian in that space, $\Delta\left(\frac{1}{r}\right)$ does not exist for $r=0$ in the sense of classical analysis but $\Delta\left(\frac{1}{r}\right)=-4\pi\delta$ in the sense of the theory of Distribution; here δ denotes a measure defined by a distribution of mass, namely a mass 1 at the origin and nothing elsewhere.

A fundamental theorem about derivatives of distributions, whose applications to the theory of Fourier series are evident, is that any convergent series of distributions may always be derived termwise.

That we should find in D′ all the classical functions of analysis, for instance those which are summable in a compact set in the sense of Lebesgue, makes obviously no difficulty. In addition to such elements, the theorem of F. Riesz just mentioned makes it clear that measures are also elements of D′. But one sees immediately that these first two categories of elements do not exhaust D′ and that this vector space yields therefore the generalization not only of the notion of function, but also of the notion of measure, or of distribution of mass. In particular, defined as a distribution, the improper function of Dirac is a measure. Its derivatives, however, are distributions but not measures, $\delta(x)$ itself is the derivative of Heaviside's function, even at $x=0$.

In the case of any Mathematician having a fairly good knowledge of the theory of vector spaces, these are notions which are simple and he will find no great difficulty even in the most sophisticated proofs of the theory. But many persons may find it hard to study such abstract notions. For the sake of those who are mainly interested in applications and problems of Mathematical Physics, Schwartz, in the first two chapters of this book explains the notion of distribution in a more elementary manner. The function space D being defined as above, a distribution T is any linear (namely additive and continuous) mapping of D into or onto the field of the complex numbers. The derivative $\frac{\partial T}{\partial x_1}(\phi)$ of the distribution $T(\phi)$ is the linear mapping $-T\left(\frac{\partial \phi}{\partial x_1}\right)$. Once the notion of derivative has been obtained, that of primitive follows easily. The second part of Chapter II shows that such an elementary theory of distributions is amply sufficient to treat extremely interesting applications of it to the theory of potentials, or to the theory of waves, or to that of meromorphic functions. The consideration of the finite part—the *partie finie* in the sense of Hadamard, in symbol Pf.—of an integral, allows Schwartz to place, if one may say so, between functions and measures, elements which he

calls "pseudofunctions"—in symbol Pf. (here the French language permits the introduction of a very suitable symbol, Pf. denoting either the initials of "partie finie" or those of "pseudo fonction"). For instance, $F(x)$ being a function of one argument which is not summable at $x=0$, and the function space D being that of functions of one real argument,

$$\text{Pf. } F(\phi) \text{ is defined as Pf. } \int_{-\infty}^{+\infty} F(x)\cdot\phi(x)\,dx.$$

If all the applications treated in Chapter II are very interesting and important, yet the mathematician will not hesitate to concentrate on the next chapter in which distributions are systematically defined as elements of the space D′.

The short space at the disposal of the reviewer forbids him to analyse the most important theorems of this chapter, Theorems XXI, XXX and XXXI. They must be considered as the most fundamental of the theory.

Chapter IV deals with direct products of distributions. They generalize the direct products of measures and their theory is mainly an extension to distributions of the known theorem of Fubini on multiple integrals. The main applications of the notion of direct product will be found in the second volume, in Chapter VI, in which, as we are told, the important operation of convolution will be applied to distributions.

Chapter V, the last of this volume, is devoted to multiplication and division of distributions. In this chapter we come across one of the main limitations and one of the greatest difficulties of the theory: a multiplication of two distributions cannot be defined in the general case. Still greater limitations are met in the problem of the division of a distribution by another. Such limitations make the theory of differential and of partial differential equations rather complicated at least as far as it is possible to form an opinion from the applications treated in this chapter. The case when the usual coefficients of the differential equations are not indefinitely derivable has to be studied with the utmost care. It seems most likely that the second volume of this treatise will reveal to us new aspects of the theory which the first volume does not allow us to suspect. It would be most interesting, in particular, to have a complete, systematic treatment of the theory of "elementary solutions" of equations of the second order. Whether the theory of Distributions furnishes new existence theorems or simply systematizes the existing theory is not yet quite clear.

In any case, this first volume, so rich in subject matter and so very well written, will certainly reward amply all those who will study it thoroughly. No one will close this first volume without eagerly looking forward to the publication of the second.

C. RACINE.

Mathematics for Technical Students

BY J. D. N. GASSON,

Book 1, Cambridge University Press, 1951, pp. xii+417.

This book aims at providing a course in mathematics for the first year of the National Certificate in Mechanical, Structural and Electrical Engineering. It consists of two parts of nearly equal size, the first devoted to Arithmetic, Algebra (up to simultaneous and quadratic equations) and the use of logarithms, while the second part deals with geometrical constructions, mensuration, graphs, and Trigonometry. The book contains at the end, logarithmic and Trigonometric four-figure tables, and an index.

There are altogether 40 chapters, each with enough material for a single lecture and a set of problems estimated to be sufficient for a first understanding of the facts together with a revision of selected points. There are about ten examples at the end of each chapter, together with a hundred revision examples at the end, divided into sets of 10 questions, and each question is accompanied by a reference to the chapter to which it is related. The examples are framed so as to illustrate the pure mathematical principles involved while there are also others which give the technical applications to Engineering problems. The printing is exceptionally good and the get up is neat and attractive.

The book should provide excellent mathematical equipment for technical students at the contemplated level.

A. N. RAO.

Cycles of Recurring Decimals

BY D. R. KAPREKAR

Published by the Author, Khare's Wada, Deolali. Price Rs. 5, pp. 8+55

In this book the author gives mainly cycles of recurring decimals for fractions of the form m/n, $m < n \leq 161$. Thus the cycles for 1/7, 2/7, 3/7, ..., 6/7 are 142857, 285714, 428571, 571428, 714285, 857142 respectively.

This simply means that $1/7 = .\dot{1}4285\dot{7}$, $2/7 = .\dot{2}8571\dot{4}$, and so on.

In the book these facts are condensed in the form :

N=7,	1–3–2–6–4–5
Cycle	1 4 2 8 5 7

showing, for example, that the cycle for 3/7 starts with 4 and that for 5/7 with 7 ; otherwise the cycle is the same in each case.

The author has taken immense pains over the work and it is of use. The book should be present in all libraries.

H. G.

Model Papers in Algebra

BY M. Y. DESHPANDE, *N. M. V. High School, Poona.*

(1949), Rs. 3/- pp. 216 (available with the author)

These papers are intended for the Matriculation and S. S. C. Students of the University of Bombay. The book is very well written, and contains good examples, including examples from real life which may be of use to the boys later. Suggestions and hints are given which make the book very useful to the students and to teachers. The printing and get up are very good.

S. M.

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BOOKS RECEIVED FOR REVIEW

RICHARD G. COOKE : *Infinite Matrices and Sequence Spaces*, Macmillan & Co., (Lond.), 1950, pp. xiv + 347, 42 sh.

L. M. MILNE-THOMSON : *Jacobian elliptic function tables*, Dover Publications, New York 19, 1950. pp. 144, $ 2–45 dollars.

STEFAN BERGMAN : *The Kernel function and conformal mapping, Mathematical Surveys, No.* 5, published by the American Math. Soc., 1950, pp. 161 + vii.

A. C. SCHAEFFER and D. C. SPENCER : *Coefficient regions for schlicht functions*, Amer. Math. Soc. Colloquium Publications, Vol. 35, $ 6–00.

N. L. JOHNSON and H. TETLEY : *Statistics, an Intermediate Text book*, Vol. II. Camb. Univ. Press, 1950, pp. 318, 20 sh. net.

A. G. GHAFFARI : *The Hodograph Method in Gas Dynamics*, Taban Press, Tehran, 1950, pp. 129.

G. C. PATNI and R. C. GAUR : *Elementary Calculus*, Ramesh Book Depot., Jaipur, 1950, pp. 232, Rs. 3.

SHAH and APTE : *Plane Analytic Geometry*, Ideal Book Service, Poona 2, 1951, pp. 252, Rs. 5-8-0.

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Publications of the Indian Mathematical Society

1. Memoir on Cubic Transformations Associated with a Desmic System, by Dr. R. Vaidyanathaswami, pp. 92—price Rs. 3.

2. Table of Partitions, by Dr. Hansraj Gupta, pp. 81, price Rs. 5.

3. The Journal of the Indian Mathematical Society.

4. The Mathematics Student.

The *Student* and the *Journal* appear quarterly. The annual subscription is Rs. 6 for each. A single issue costs Rs. 1-8-0.

Applications for membership and all business communications should be addressed to Prof. S. Mahadeva Iyer, M.A., L.T., 2, Bhaskarapuram, Mylapore, Madras.

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